Contact Structures On 3-Manifolds And Their Classification In The Overtwisted Case*

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1 Introduction

The aim of this essay is to give an introduction to the theory of contact structures on 3-manifolds and their classification in the overtwisted case. The first two sections give definitions and basic properties. The third sections states the main theorem. The next three describe tools we need to give a proof; the proof itself is finally contained in the last section.

2 Contact structures

Let M always denote a 3-dimensional manifold, and let TM be its tangential bundle. Consider a plane distribution ξ in TM. Locally it can be written as the kernel of a 1-form α. If α satisfies the condition

$$\alpha \wedge d\alpha \neq 0 \text{ everywhere,}$$

we call ξ a *contact structure*. This condition is independent of the choice of α for a ξ.

Remark 2.1. The integrability condition for plane fields ξ on M can be written as $\alpha \wedge d\alpha = 0$. So contact structures can be described as maximally non-integrable plane fields, which will be important for the foliations later.

Definition 2.2. Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called *contactomorphic* if there is a diffeomorphism $f : M_1 \to M_2$ with $Tf(\xi_1) = \xi_2$, where $Tf : TM_1 \to TM_2$ denotes the differential of f. If $\xi_1 = \ker \alpha_1, \xi_2 = \ker \alpha_2$, this is equivalent to the existence of a nowhere zero function $\lambda : M_1 \to \mathbb{R}$ such that $f^*\alpha_2 = \lambda\alpha_1$.

Example 2.3. *$\alpha_1 = dz + xdy$ and $\alpha_2 = dz + \rho^2 d\phi$ are two contact structures on $\mathbb{R}^3$ which are defined in cartesian (x, y, z) and cylindric (ρ, ϕ, z) coordinates respectively. They are both contactomorphic: Choose*

$$f(x, y, z) \quad = \quad ((x + y)/2, (y - x)/2, z + xy/2)\,,$$

then $f^*\alpha_2 = \alpha_1$. This contact structure is called the standard contact structure *ζ_0.*

For every contact structure ξ, the form α defines a volume form $\alpha \wedge d\alpha$ on M, which is independent of the sign of α. So ξ gives us an orientation on M. We will always consider oriented manifolds M on which the orientation coincides with the one we get from the contact structure.

The following two theorems, which are familiar from symplectic geometry, hold as well in contact geometry:

Theorem 2.4 (Gray Stability)**.** *Let ξ_t, $t \in [0, 1]$ be a smooth family of contact structures on a closed manifold M. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that*

$$T\psi_t(\xi_0) \quad = \quad \xi_t \quad \textit{for each} \quad t \in [0, 1].$$

Theorem 2.5 (Darboux)**.** *Let α be a contact form on the manifold M and p a point on M. Then there are coordinates x, y, z on a neighbourhood $U \subset M$ such that*

$$\alpha_{|U} = dz + xdy$$

3 Overtwisted Contact Structures

Definition 3.1. The *standard overtwisted structure* on $\mathbb{R}^3$ is given by $h_1dz + h_2d\phi$ in cylindric coordinates (ρ, ϕ, z), where $h_1(\rho)$ and $h_2(\rho)$ are defined by the following diagram:

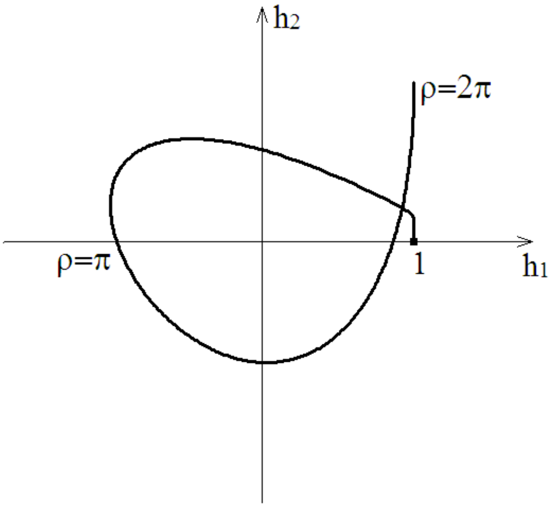

Figure 1: function of Lutz twist

By expanding $\alpha \wedge d\alpha$ for $\alpha = h_1dz + h_2d\phi$, we see that α is a contact structure iff the tangent of $(h_1(\rho), h_2(\rho))$ in $\mathbb{R}^2$ never equals a ray through the origin. This is obviously not the case in the picture.

Let $D^2(\pi)$ now be the disc of radius π and $\theta(\rho)$ a smooth function with $\theta(\pi) = 0$, $\theta(\rho) > 0$ for $0 \leq \rho < \pi$, and $\theta'(\rho) = 0$ iff $\rho = 0$. Consider the embedding:

$$\begin{array}{ccc} D^2(\pi) & \overset{f}{\to} & \mathbb{R} \times D^2(\pi) \\ (\rho, \phi) & \mapsto & (\theta(\rho), \rho, \phi) \end{array}$$

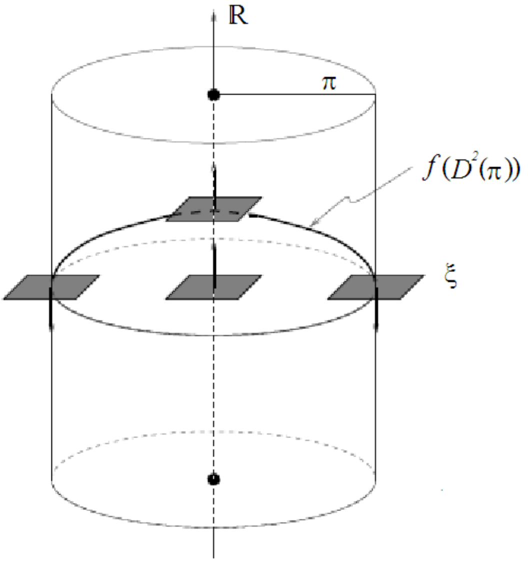

Figure 2: overtwisted disc in cylinder (from [3], with different notation)

It defines a 1-form on $D^2(\pi)$:

$$f^* \left(h_1(\rho)dz + h_2(\rho)d\phi\right) \quad = \quad h_1(\rho)\theta'(\rho)d\rho + h_2(\rho)d\phi$$

Its kernel gives us a singular oriented line distribution on $D^2(\pi)$, which integrates to a one-dimensional foliation of $D^2(\pi)$. The only singular point (where both the coefficients of dz and $d\phi$ vanish) is zero. As $h_1(\rho)$ tends to zero as ρ goes to π, we get the boundary of $D^2(\pi)$ as limit cycle of the foliation. The whole disc looks like this:

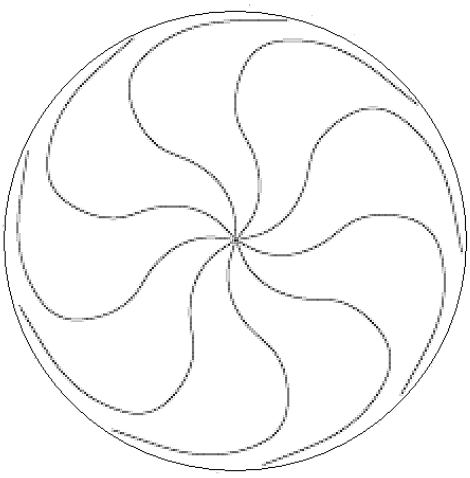

Figure 3: overtwisted disc (from [3])

Because of this appearance we call this kind of disc *overtwisted.* A contact structure (M, ξ) is called (standardly) overtwisted if there is a contact embedding of a neighbourhood of the standard overtwisted disc into (M, ξ).

Overtwisted discs can be produced in every contact structure by *Lutz twists.*

3.1 Lutz twist

We consider an arbitrary contact manifold (M, ξ) and a knot K in M which is transverse to ξ. Using a example 2.33 from [3], we can identify a neighbourhood of K with $S^1 \times D^2(2\pi)$ with the standard contact structure induced by $dz + \rho^2 d\phi$, where z now denotes the S^1-direction. We can now spoil this structure and make it overtwisted without changing the homotopy type of the plane distribution by using a *Lutz twist.* Therefore we replace the standard structure on $S^1 \times D^2(2\pi)$ by the overtwisted structure. It is important that we have

$$h_1(\rho) = 1 \quad \text{and} \quad h_2(\rho) = \rho^2 \quad \text{for} \quad \rho \in [0, \varepsilon] \cup [2\pi - \varepsilon, 2\pi],$$

so that the new structure is smooth. Now we have to find a homotopy; therefore we choose $\left(h_1^t(\rho), h_2^t(\rho)\right)$ with $\rho \in [0, 2\pi]$ and $t \in [0, 1]$ which satisfies:

1. $h_1^0 \equiv 1, h_2^0(\rho) = \rho^2$
2. $h_1^1 \equiv h_1, h_2^1 \equiv h_2$
3. $h_i^t(\rho) = h_i(\rho) \quad \text{for} \quad \rho \in [0, \varepsilon] \cup [2\pi - \varepsilon, 2\pi].$

As this homopoty will have a point where both h_1^t and h_2^t vanish, we have to add a component in the ρ-direction. For this let $\chi : [0, 2\pi] \to \mathbb{R}$ be a smooth function which is identically zero near $\rho = 0$ and $\rho = 2\pi$ and satisfies $\chi(\rho) > 0$ in $[\varepsilon, 2\pi - \varepsilon]$. Then

$$\alpha_t = t(1-t)\chi(\rho) d\rho + h_1^t(\rho) dz + h_2^t(\rho) d\phi$$

is a homotopy from $\alpha_0 = dz + \rho^2 d\phi$ to $\alpha_1 = h_1(\rho) dz + h_2(\rho) d\phi$ through non-zero 1-forms. As we do not change anything near $\rho = 2\pi$, we get a homotopy of the plane fields.

Theorem 3.2. *Let ξ be a contact structure on M. By a Lutz twist along any transversely embedded circle one obtains an overtwisted contact structure ξ' which is homotopic to ξ as a plane distribution.*

4 The Classification Theorem

Definition 4.1. Let $\mathrm{Cont}^{\mathrm{ot}}(M)$ be the space of standardly overtwisted contact structures and $\mathrm{Distr}(M)$ be the space of all plane distributions on M. Both will be equipped with the C^∞-topology (see e.g. weak topology in [5], p. 34)

Theorem 4.2 (Classification). *The inclusion* $j : \mathrm{Cont}^{\mathrm{ot}}(M) \to \mathrm{Distr}(M)$ *is a homotopy equivalence for* M *closed, oriented and connected.*

Using Gray-stability we get the immediate corollary:

Corollary 4.3. *Two overtwisted contact structures are isotopic iff they are homotopic as plane distributions.*

Let us now state an a bit more general version of this theorem, which will fit better for our proof:

Theorem 4.4. *Let* M *be a compact 3-manifold and let* $A, A \subset M$*, be a closed subset such that* $M \backslash A$ *is connected. Let* K *be a compact space and* L*,* $L \subset K$*, a closed subspace. Let* ξ_t*,* $t \in K$*, be a family of plane distributions which are contact everywhere for* $t \in L$ *and are contact near* A *for* $t \in K$*. Suppose there exists an embedded 2-disc* $\Delta \subset M \backslash A$ *such that* ξ_t *is contact near* Δ *and* (Δ, ξ_t) *is equivalent to the standard overtwisted disc* (Δ, ζ_0) *for all* $t \in K$*. Then there exists a family* ξ'_t*,* $t \in K$*, of contact structures on* M *such that* ξ'_t *coincides with* ξ_t *near* A *for* $t \in K$ *and coincides with* ξ_t *everywhere for* $t \in L$*. Moreover* $\xi'_t, t \in K$ *can be connected with* ξ_t*,* $t \in K$ *by a fixed on* $A \times K \cup M \times L$ *homotopy through families of distributions.*

To see that this theorem implies theorem 4.2 we set $L = \emptyset$, $K = p$ and $A = \emptyset$. To produce an overtwisted disc in ξ_t, we notice that every plane distribution is locally trivial, so it can be perturbed into the standard overtwisted structure and then Lutz twisted. The theorem now implies that every plane distribution can be connected with a contact structure by a homotopy of plane distributions. This implies the classification theorem.

Our general strategy is to transmute the plane distribution into a contact distribution by first changing it near the 2-skeleton of a simplicial complex and then extend this perturbation to the inner “balls”. For this we first discuss foliations of 2-spheres.

5 Foliations Of Surfaces

If we embed a 2-dimensional compact surface S into a contact 3-manifold, the tangent bundle TM and the plane distribution ξ intersect each other in planes and lines. Points with intersection planes have to be isolated because ξ is non-integrable. As S is compact, there can only be finitely many such points. Because M and ξ are (locally) oriented, we get an oriented singular line distribution which integrates to a 1-dimensional singular foliation of S.

5.1 Foliations on S^2

Definition 5.1 (Simple and almost horizontal foliations). A 1-dimensional oriented foliation on S^2 with two singular points of focus type is said to be *simple* if all its limit cycles are isolated and placed on parallels between the two focuses (see Fig 4) and if one of its focuses is stable and the other is unstable. We will call the focuses north and south poles respectively.

A simple foliation $\mathfrak{F}$ on S^2 is called *almost horizontal* if there is a transversal $\mathfrak{t}$ to $\mathfrak{F}$ connecting its poles.

If an oriented contact structure ξ, defined in a neighbourhood of a sphere S embedded in a 3-manifold, generates a simple or almost horizontal foliation S_ξ on S, then we say that the contact structure itself is simple or almost horizontal near S.

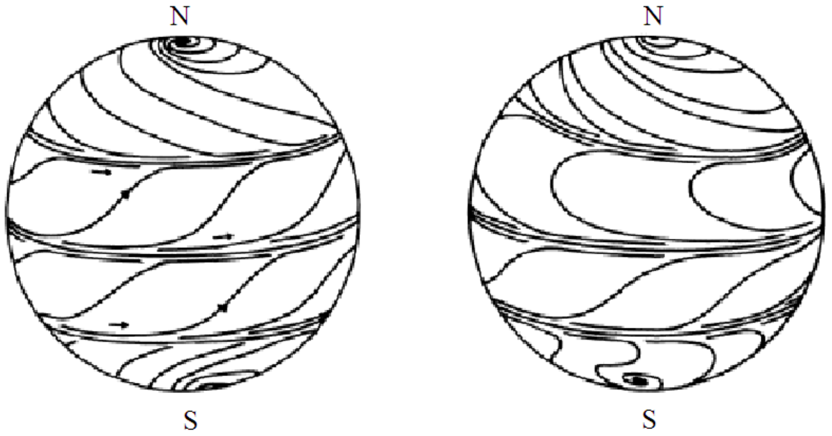

Figure 4: Simple and almost horizontal foliations of S^2 (from [1])

If $\mathfrak{F}$ is an almost horizontal foliation with a transversal $\mathfrak{t}$ from north- to south pole, we can define a map $f : \mathfrak{t} \to \mathfrak{t}$ through the holonomy: At every point of $\mathfrak{t}$ we follow the intersecting (oriented) leaf until it intersects $\mathfrak{t}$ again. This map is obviously bijective and is also differentiable (see [2], p. 194), so it is a diffeomorphism of the transversal. Since the transversal $\mathfrak{t}$ is a path, we also get a diffeomorphism $h(\mathfrak{F}) : I \to I$ which is defined up to conjugacy. For a diffeomorphism $h : I \to I$, we denote by $\mathfrak{F}(h)$ an almost horizontal foliation with $h(\mathfrak{F}(h)) = h$.

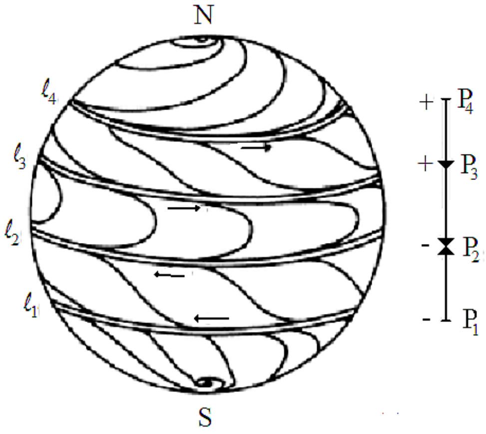

Figure 5: Almost horizontal foliation of S^2 with diagram (from [1])

We now want to describe the topological type of a simple foliation $\mathfrak{F}$ by easier data. For this we define the diagram:

Definition 5.2 (diagram of simple foliation)**.** A limit cycle of a simple foliation $\mathfrak{F}$ divides S^2 into two hemispheres: the lower one (containing the south pole) and the upper one. We call the limit cycle *positive* if it is oriented as the boundary of the lower hemisphere and negative in the opposite case. We order the limit cycles of $\mathfrak{F}$ in the direction from the south to the north poles $\mathfrak{l}_1, \dots, \mathfrak{l}_p$. The *diagram* $\mathfrak{D}(\mathfrak{F})$ of the simple foliation $\mathfrak{F}$ consists of p points $q_1, \dots, q_p$ on a line with the following additional structure: to each point q_i is assigned the sign of the limit cycle $\mathfrak{l}_i$; each interval $q_i q_{i+1}$ is oriented from q_i to q_{i+1} (resp. from q_{i+1} to q_i) if the cycle $\mathfrak{l}_i$ is unstable (resp. stable) for the foliation $\mathfrak{F}$ restricted to the band between $\mathfrak{l}_i$ and $\mathfrak{l}_{i+1}$ (see Fig. 5).

Now we get as a corollary of the Poincaré-Bendixson theorem (see [2], p. 231)

Theorem 5.3. *The topological type of a simple foliation* $\mathfrak{F}$ *is uniquely defined by its diagram* $\mathfrak{D}(\mathfrak{F})$*.*

Proof. We only have two critical points and no limit cycles through them. So every leaf above $\mathfrak{l}_p$ has to come from $\mathfrak{l}_p$ and go to the north pole, analogous for the south pole. Between two circles all the leaves have to come from one and go to the other one or the other way round. This is specified by the diagram. We choose closed curves between the limit cycles and make them transversal to the foliation (see below how you do that). Then we notice that every leaf between two limit cycles crosses the closed curve exactly once, since two intersection points would have different orientations and so a tangent vector in between. So if we have two foliations with the same diagram, we can choose transversals $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in both of them and map the leaf through $\mathfrak{t}_1(x)$ onto the leaf through $\mathfrak{t}_2(x)$ for every $x \in S^1$. This gives the required homeomorphism. ☐

We should mention another construction with which we can combine to foliated surfaces to one:

Definition 5.4 (Connected sums of simple foliations)**.** For two simple foliations $\mathfrak{F}$ on S and $\mathfrak{F}'$ on S' we define the *connected sum* $\mathfrak{F}\#\mathfrak{F}'$ on $S\#S'$ as follows: Take small discs $D \subset S$ and $D' \subset S'$ which contain the north pole of $\mathfrak{F}$ and the south pole of $\mathfrak{F}'$ respectively. The boundaries of D and D' should be chosen to be transverse to $\mathfrak{F}$ and $\mathfrak{F}'$. Now attach $S'\backslash D'$ to $S\backslash D$ along ∂D and $\partial D'$ and smooth the resulting foliation on $S\#S'$.

In this way $\mathfrak{F}\#\mathfrak{F}'$ is defined up to homeomorphism.

5.2 Extendabilty of contact structures

Lemma 5.5. *Let ξ be a simple contact structure near the boundary $S = \partial B$ of a 3-ball B. The extendability of ξ as a contact structure to B depends only on the topological type of the foliation S_ξ.*

Proof. Let ξ and ξ' be two contact structures defined near S which induce foliations of the same topological type. Without loss of generality we can assume that the poles and limit cycles induced by ξ and ξ' coincide.

Let $\mathfrak{t}_{\mathfrak{S}}$ and $\mathfrak{t}_{\mathfrak{N}}$ be closed transversals to ξ which are close to the poles of S. Let L be the union of the limit cycles and the transversals $\mathfrak{t}_{\mathfrak{S}}$ and $\mathfrak{t}_{\mathfrak{N}}$. Denote by N a small tubular neighbourhood of L.

We look at a neighbourhood H of $S\backslash N$. We can define a diffeomorphism $g : H \to H$ which is a contactomorphism for the germs ξ and ξ' of the contact structure. For the definition look at a component between two limit cycles which looks like $S^1 \times I$. Choose two closed curves $\mathfrak{k}$ and $\mathfrak{k}'$ parallel to S^1 and make them transversal to the contact structures ξ and ξ' respectively (see below how to do this). They intersect every leaf exactly once since two intersection points would have different orientations and so a tangent vector in between. Now we can map the leaf through $\mathfrak{k}(x)$ onto the leaf through $\mathfrak{k}'(x)$ for every $x \in S^1$. This defines g on $S\backslash N$ (we do the same things near the poles).

Now we use Theorem 2.39 from [3]:

Let S_i be (compact parts of) closed surfaces in contact 3-manifolds (M_i, ξ_i), $i = 0, 1$ (with ξ_i coorientable), and $\phi : S_0 \to S_1$ a diffeomorphism with $\phi(S_{0,\xi_0}) = S_{1,\xi_1}$ as oriented characteristic foliations. Then there exists a contactomorphism $\psi : \mathcal{N}(S_0) \to \mathcal{N}(S_1)$ of suitable neighbourhoods $\mathcal{N}(S_i)$ of S_i with $\psi(S_0) = S_1$ and such that $\psi|_{S_0}$ is isotopic to ϕ via an isotopy preserving the characteristic foliation.

If we apply the theorem onto the components of $S\backslash N$, we get the map we wanted.

Since we can choose N discretionary small and as the derivate of the leaves approaches zero near L, we can extend g to a diffeomorphism g' from $S \to S$ which is constant on L and such that the foliation $\mathfrak{F} = g'(S_{\xi'})$ is $\mathcal{C}^1$-close to S_ξ. Near the poles the $\mathcal{C}^1$-closeness depends on the choice of $\mathfrak{t}_{\mathfrak{S}}$ and $\mathfrak{t}_{\mathfrak{N}}$.

Let $\tilde{B}$, $B \subset \tilde{B}$, be a larger ball on which the structure ξ is still defined. Let us show that there exists an embedding $\tilde{g} : S \to \tilde{B}$ which is $\mathcal{C}^0$-close to the inclusion $S \subset B$, is the identity on L and outside a small neighbourhood of L and such that the foliation $(\tilde{g}(S))_\xi$ is diffeomorphic to $\mathfrak{F}$.

Near each limit cycle $\mathfrak{l}$ the contact structure ξ is equivalent to the standard overtwisted structure ζ_1 which can be written as $\alpha = \cos\rho dz + \rho \sin\rho d\phi$ near the circle $C = \{\rho = \pi, z = 0\}$, because contact structures are unique near Legendrian curves (see [3], example 2.29).

A leaf $\mathfrak{l}$ of $\mathfrak{F}$ through a point $P = (\rho_0, \phi_0, z_0)$ can be locally written as $(f_1(\phi), \phi, f(\phi))$. We write $\tilde{g}(P) = (h(P), \phi_0, z_0)$ and want to calculate h. As $\mathfrak{l}$ should be mapped onto a leaf of the foliation of $\tilde{g}(S)$, we must have $\alpha(T\tilde{g}(T\mathfrak{l}_P)) = 0$, where $T\mathfrak{l}_P$ is the tangent vector to $\mathfrak{l}$ at P. We get

$$\begin{aligned} \alpha(T\tilde{g}(T\mathfrak{l}_P) &= \alpha\left(\begin{pmatrix} \frac{\partial h}{\partial \rho} & \frac{\partial h}{\partial \phi} & \frac{\partial h}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1'(\phi) \\ 1 \\ f'(\phi) \end{pmatrix}\right) \\ &= \cos h(P) \cdot f'(\phi) + h(P)\sin h(P), \end{aligned}$$

which leads to

$$f'(\phi) = -h(P)\tan(h(P)).$$

This has a unique solution $h(P)$.

Near the transversals $\mathfrak{t}_{\mathfrak{N}}$ and $\mathfrak{t}_{\mathfrak{S}}$ we have a standard contact structure and can define $\tilde{g}$ in similar manner. If $\mathfrak{l}$ is also a leaf of the foliation by ξ, h is just the projection on the first parameter, so $\tilde{g}$ is the identity. So we can extend $\tilde{g}$ to the whole of S and it satisfies the conditions (The $\mathcal{C}^0$-closeness is true because $f'(\phi)$ becomes arbitrary small when N gets smaller, so $h(P)$ gets close to π.).

To finish the proof, note that the diffeomorphism of foliations $\tilde{g}g' : S \to \tilde{g}_\xi(S))$ can be covered by contact diffeomorphism of germs ξ' and ξ on S and $\tilde{g}(S)$. Because the structure $\xi|_{\tilde{g}(S)}$ is already extended to the ball, the same is true for $\xi'|_S$. □

5.3 The contact structure near the overtwisted disc

We look at the neighbourhood of the overtwisted disc in the standard overtwisted structure ζ_1 on $\mathbb{R}^3$. If we have a curve in the (z, ρ)-plane, we can consider the surface of revolution which we get by turning around the z-axis. Convex surfaces of revolution which contain the overtwisted disc always meet the π-cylinder twice. So we get a foliation $\mathfrak{H}$ with two limit cycles looking like this:

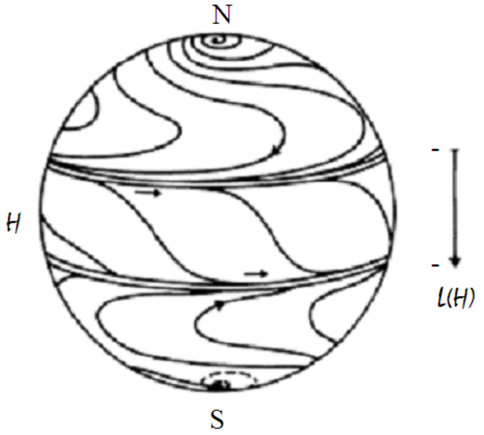

Figure 6: foliation of convex surface of revolution

Since every (standardly) overtwisted structure contains a neighbourhood of the overtwisted disc which is contactomorphic to the standard overtwisted structure, we can always find a surface in the neighbourhood of the overtwisted disc which has the foliation $\mathfrak{H}$.

6 Existence Of Transversals

For a given curve $\gamma : I \to M$ we want to find a $\mathcal{C}^0$-close curve with same start- and endpoint which is Legendrian, i.e. is everywhere tangent to the contact structure. We can reduce the problem to the case $M = \mathbb{R}^3$ with standard contact structure by covering the image of γ by finitely many Darboux charts. If we can to a pertubation in any of the charts, we can put all these together to a pertubation of γ on M. First we consider the so-called Lagrangian projection on the first two coordinates:

Lemma 6.1. *Let $\gamma : (a,b) \to (\mathbb{R}^3, \xi)$ be a Legendrian immersion. Then its Lagrangrian projection $\gamma_L(t) = (x(t), y(t))$ is also an immersed curve. The curve γ is recovered from γ_L via*

$$z(t_1) \quad = \quad z(t_0) - \int_{t_0}^{t_1} x dy.$$

The curve γ is embedded iff every loop in γ_L encloses non-zero oriented area.

Any immersed cure in the (x,y)-plane is the Lagrangian projection of a Legendrian curve in $\mathbb{R}^3$, unique up to translation in the z-direction.

Proof. The Lengendrian condition in the standard contact structure $dz + xdy$ is given by $\dot{z} + x\dot{y} = 0$ ($\dot{z}$ means the derivative of z with respect to t). So if $\dot{y} = 0$ then $\dot{z} = 0$, and hence, since γ is an immersion, $\dot{x} \neq 0$. So γ_L is an immersion.

The formula for z follows by integrating the Legendrian condition. If we just look at a closed part of the curve γ_L, we know by Stokes (denoting the piece of the curve by l and the enclosed compact set by A):

$$\int_A dx \wedge dy \quad = \quad \int_l xdy$$

As the difference between the two z-coordinates of the start- and endpoint of l is given by the second integral, γ meets itself exactly if the area of A vanishes. $\square$

Theorem 6.2. *Let* $\gamma : I \to (\mathbb{R}^3, \xi)$ *be an embedding. Then* γ *can be* $\mathcal{C}^0$*-approximated by a Legendrian curve.*

Proof. In order to find a $\mathcal{C}^0$-approximation of γ by a Legendrian curve, one only has to approximate its Lagrangian projection γ_L by an immersed curve of which the area integral

$$z(t_0) - \int_{t_0}^{t} xdy$$

lies as close to the original $z(t)$ as one wishes. This is possible by introducing small loops, which increase or decrease the area function depending on the orientation and so on the direction in which the curve intersects itself. Since we can also produce overlapping loops, we can increase or decrease the area to any finite number in an arbitrarily small neighbourhood of our originial curve. So we get the required $\mathcal{C}^0$ approximation of γ_L while simutainiously approximating the z-coordinate. The following picture may illustrate this:

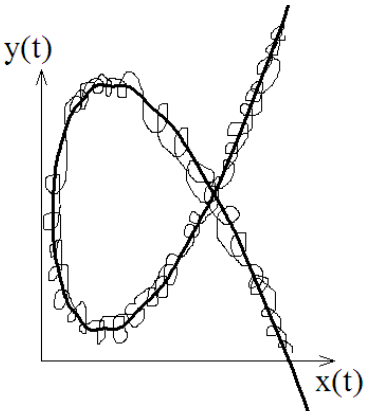

Figure 7: $\mathcal{C}^0$-approximation

The curve in $\mathbb{R}^3$ we get from our two-dimensional approximation is still embedded because at every self-intersection point in $\mathbb{R}^2$ we enclose non-zero area (by sensible choice). So we have found our Legendrian curve $\mathcal{C}^0$-close to γ. $\square$

Theorem 6.3. *For every curve* γ *as above you can find a transverse (to* ξ*)* $\mathcal{C}^0$ *approximation.*

Proof. By the theorem above we may already assume that our curve γ is Legendrian. But near a Legendrian submanifold, there is always a neighbourhood $I \times D^2$ on which the contact structure has the form $\cos\theta dx - \sin\theta dy$ for coordinates (θ, x, y) (see [3]). γ is given by $I \times 0$. Now we explicitely define γ' by

$$\gamma'(t) \quad = \quad (t, \delta\sin t, \delta\cos t)$$

This is transverse for every $\delta > 0$ because the form evaluates to δ. Since δ can be arbitrarily small, we get a $\mathcal{C}^0$ approximation of γ. ☐

Theorem 6.4. *For every family γ_t of curves ($t \in K$, K compact) we can find a family γ'_t of transverse to ξ curves which are $\mathcal{C}^0$ close (for each t).*

Proof. The construction of Legendrian and then of transverse curves out of γ does continuously depend on the choice of γ. ☐

7 Construction Near A 2-Skeleton

We now look a simplicial complexes representing our manifold (locally).

Definition 7.1. A compact simplicial complex $P \subset \mathbb{R}^3$ is called *general* if no two faces or a face and an edge with a mutual vertex are contained in one plane. We denote:

- $\alpha(P)$ is the minimal angle between non-incident 1- or 2-simplices which have a mutual vertex.
- $d(P)$ is the maximal diameter of a simplex of P.
- $\delta(P)$ is the minimal distance between two 0-, 1- or 2-simpices without mutual vertices.

Theorem 7.2. *There exists a sequence of general subdivisions P_i of P such that $d(P_i) \to 0$ while $\delta(P_i)/d(P_i)$ and $\alpha(P_i)$ are bounded below by a positive $\varepsilon > 0$.*

Proof. We want to decrease $d(P)$ until it is smaller than a given ε'. Since you can divide every simplex into cubes (to be a cube should just mean the combinatorial property of having 8 vertices which are connected by the right number of edges - not equality of lengthes or something like that), you can divide our simplicial complex into cubes. Now subsequently divide these cubes into $3 \times 3 \times 3$ cubes, until the diameter is smaller than $\varepsilon'/2$. Now triangulate without creating new vertices to get P_i. The quotient $\delta(P_i)/d(P)$ and the angle α does not depend on how fine we triangulate, especially it stays greater than a chosen 2ε. Now move every face in general position; this can be done without changing any of the variables more than $\max\{\varepsilon'/4, \varepsilon/2\}$. ☐

We will now define the *norm* of a plane distribution. This is the derivative of the change of direction, which is the angle with which in turns. We will try to control this norm to construct $\mathcal{C}^0$-close distributions.

Definition 7.3 (Norm)**.** An oriented plane distribution ξ on a compact $A \subset \mathbb{R}^3$ defines the Gauß mapping $G_\xi : A \to S^2$ which is the normal vector. The number $||\xi|| = \max_{x \in A} ||Tg(x)||$ is called the *norm* of the distribution ξ.

Theorem 7.4. *Let $\mathfrak{F}$ be a two-dimensional foliation on the closure $\bar{U}$ of a bounded domain $U \subset \mathbb{R}^3$ with simply connected leaves. Let K be a compact space and L be a closed subset. Let ξ_t, $t \in K$, be a family of plane distributions on $\bar{U}$ transversal to $\mathfrak{F}$. Suppose that ξ_t is contact near a closed $A \subset U$ for all $t \in K$ and is contact everywhere for $t \in L$. Suppose that for any leaf V of $\mathfrak{F}$ and any leaf $\mathfrak{l}$ of the induced foliation V_{ξ_t}, $t \in K$, on the leaves of $\mathfrak{F}$ we have $\pi_1(\mathfrak{l}, A \cap \mathfrak{l}) = 0$. Then there exists a family ξ'_t, $t \in K$, of contact structures on $\bar{U}$ such that ξ'_t coincides with ξ_t on A for all $t \in K$ and coincides with ξ everywhere for $t \in L$.*

Suppose further that there are constants c_0, c_1 and $\varepsilon > 0$ so that the norm of the plane distribution tangent to $\mathfrak{F}$ is less than c_0, the minimal angle between ξ_t and $\mathfrak{F}$ is greater than ε, and the diameter of U is less than c_1. Then there are constants C and D which only depend on c_0, c_1 and ε so that $||\xi'_t|| \leq C||\xi_t|| + D$.

Proof. For $t \in K$ let $\mathfrak{G}_t$ be the one-dimensional tangent to ξ_t foliation on $\bar{U}$ formed by all leaves of foliations V_{ξ_t}, when V runs through all leaves of $\mathfrak{F}$. There can be no circle leaves in $\mathfrak{G}_t$ since this would imply a circle leaf in a 2-dimensional leaf V of $\mathfrak{F}$; as all the leaves of $\mathfrak{F}$ are simply connected, we would get a critical point in the foliation of V inside the circle. This is not possible since $\mathfrak{F}$ and ξ_t are transverse. As we have no circles and not critical points in $\mathfrak{G}_t$, we cannot have leaves which stay in a compact set for infinite time (Poincaré-Bendixson). So every leaf enters and leaves $\bar{U}$ at one point.

The foliation $\mathfrak{G}_t$ can be seen as introduced by a unit vector field with the right orientation in $T\mathfrak{F} \cap \xi_t$. In this way we can parametrize every leaf $\mathfrak{l}$ by $s \in [0, s(\mathfrak{l})]$ between the point where it enters and leaves $\bar{U}$.

In $\mathbb{R}^3$ with the standard structure ζ_0 defined by the 1-form $dz - ydx$, we denote by $\mathfrak{L}$ the Legendrian foliation by lines parallel to the y-axis. For every point P in the boundary for which a leaf enters we consider a disc-shaped neighbourhood D and a the closed subset D' which consists of all the points in which leaves enter. We embed D' into the (x, z)-plane and map each leaf starting in D' onto the leaf of $\mathfrak{L}$ which goes through the same point in the (x, z)-plane where the y-coordinate equals s in the parametrisation. We name the set of all leaves through D' by U_t^P. This map (we call it h) is injective and continuous. The set U_t^P is closed: If we consider a sequence of points (x_i, s_i) in U_t^P, $x_i \in D'$, which converges against a point (x, s), we can choose a subsequence of the x_i which converges in the compact set D'. So $x \in D'$ and $(x, s) \in U_t^P$. Since we now have a continuous injective map from a compact space to a Hausdorff space, we have a homeomorphism.

Since we can cover the boundary with finitely many discs by compactness, we can cover all leaves by finitely many subsets U_t^i of this kind. So there is a cover

$$\bar{U} \quad = \quad \bigcup_{i=1}^{N} U_t^i.$$

We can choose it continuously depending on $t \in K$. By the above construction we get maps $h_t^i : U_t^i \to \mathbb{R}^3$ with $(h_t^i)^* \mathfrak{L} = G_t|_{U_t^i}$. The maps h_t^i also induce a contact structure on $\mathbb{R}^3$; since we have Legendrian curves in the y-direction, the contact structure is given by $\alpha = P_t(x,y,z)dx + Q_t(x,y,z)dz$. When a point (x,y,z) moves along a leaf $\mathfrak{l}$ of $\mathfrak{L}$ the point $(P_t(x,y,z), Q_t(x,y,z))$ draws a curve $\mathfrak{l}'$ in $\mathbb{R}^2$. Writing down the contact condition gives immediately that α is exactly contact if the curve in $\mathbb{R}^2$ is never tangent to rays from the origin.

Since we have the condition $\pi_1(\mathfrak{l}, A \cap \mathfrak{l}) = 0$ for all the leaves, we know that every leaf hits A at most once.

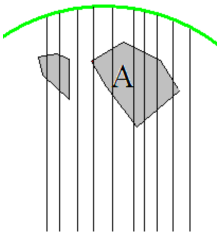

Figure 8: foliation in $\mathbb{R}^3$ with given A

A splits our leaves into intervals which have at most one end in A. We can now change the plane distribution away from the part near A to a contact structure ξ_t'. We do that by continueing the curve in the direction in which turns around the origin when it leaves A. We will bound the speed of the curve (and so the norm of ξ_t') by the speed of the original curve. This original speed is controlled by $||\xi_t||$ and the angle between $\mathfrak{F}$ and ξ_t.

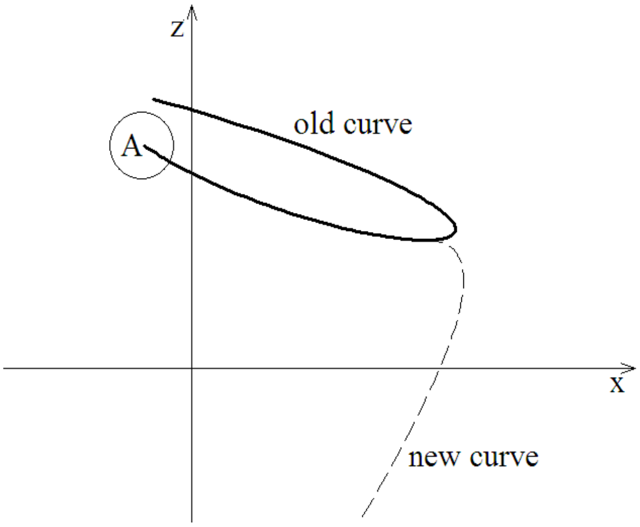

Figure 9: change of ξ_t along one leaf

So we find the pertubation in U_t^i. Now we do this step by step for $i = 1$ to $i = N$; after every step we extend A by the set on which we already worked. ☐

Now we want to use this theorem to find a contact structure near the 2-skeleton of a general simplicial complex:

Lemma 7.5. *Let K be a compact space and L a closed subset. Let ξ_t, $t \in K$, be a family of distributions defined near a compact $B \subset \mathbb{R}^3$ which are contact near a closed $A \subset B$ for $t \in K$ and are contact everywhere for $t \in L$. Then there exists a general simplicial complex $B \subset P$ and a family of distributions ξ'_t, $t \in K$, with the following properties:*

1. *ξ'_t is $\mathcal{C}^0$-close to ξ_t, $t \in K$;*
2. *ξ'_t coincides with ξ_t on A for $t \in K$ and everywhere for $t \in L$;*
3. *there exists $\varepsilon > 0$ depending only on $\alpha(P)$ and $\delta(P)/d(P)$ such that ξ'_t, $t \in K$, is contact in $\varepsilon \cdot d(P)$-neighbourhood of the 2-skeleton of P;*
4. *$||\xi'_t|| \leq C||\xi_t|| + D$, $t \in K$, for universal constants C and D.*

Proof. We can choose a general simplicial complex $B \subset P$ and its subcomplex $A \subset Q$ such that for all $t \in K$ the distribution ξ_t is still defined on P and contact on Q. We can suppose moreover that each simplex of $P \backslash Q$ has at most one face belonging to Q.

We write $N = \max_{t \in K} ||\xi_t|| + 1$ and leave out the parameter P in δ, d and α. According to theorem 7.2 we can make d arbitrarily small without changing α and δ/d. Denote by $\gamma(\pi_1, \pi_2)$ the angle between two planes or lines π_1 and π_2

in $\mathbb{R}^3$. Consider a covering $\cup_1^p K_i$ of K such that for each $i = 1, \dots, p$ and for any $t, t' \in K_i$ and $x \in P$ we have:

$$\gamma\left(\xi_t(x), \xi_{t'}(x)\right) \quad < \quad \frac{\alpha}{16} \tag{1}$$

We can do our perturbation on each K_i from 1 to p and always extend L. So we can WLOG assume that the condition holds for the whole of K.

We call a 1- or 2-simplex $\sigma \in P \backslash Q$ *special* if $\gamma\left(\sigma, \xi_t(x)\right) < \frac{\alpha}{4}$ for some $t \in K$, $x \in \sigma$.

If we choose $d < \frac{\alpha}{16N}$, then for any *non-special* simplex σ, any point x belonging to 3-simplex incident to σ and any $t \in K$ we have (for any $y \in \sigma$):

$$\gamma\left(\sigma, \xi_t(x)\right) > \gamma\left(\sigma, \xi_t(y)\right) - \gamma(\xi_t(y), \xi_t(x)) > \frac{\alpha}{4} - \frac{\alpha}{16} > \frac{\alpha}{8} \tag{2}$$

Near each simplex σ of $P \backslash Q$ of dimension ≤ 2 consider a foliation $\mathfrak{F}_\sigma$ by planes which are perpendicular to $\xi_t(x)$ for some $t \in K$, $x \in \sigma$, parallel to σ if $\dim \sigma = 1$ and perpendicular to σ if $\dim \sigma = 2$.

Given foliations $\mathfrak{F}_\sigma$, we will construct the required pertubation ξ_t' of ξ_t, $t \in K$, in three steps using the last theorem. First we notice that all special simplices are isolated. Assuming that σ and σ' are adjacent, we get the inequality (if $\gamma(\xi_t(x), \sigma), \gamma(\xi_{t'}(x'), \sigma') < \frac{\alpha}{4}$):

$$\begin{aligned} \gamma(\sigma, \sigma') \quad &< \quad \gamma(\sigma, \xi_t(x)) + \gamma(\xi_t(x), \xi_{t'}(x)) + \gamma(\xi_{t'}(x), \xi_{t'}(x')) + \gamma(\xi_{t'}(x'), \sigma') \\ &< \quad \frac{\alpha}{4} + \frac{\alpha}{16} + \frac{\alpha}{16} + \frac{\alpha}{4} < \alpha \end{aligned}$$

This is not possible since all adjacent simplices have an angle greater than equal to α. So we can change ξ_t in a neighbourhood of all special simplices and all vertices which do not belong to special simplices by the above theorem. We do not have to consider the condition about π_1, since every simplex has at most one face in common with Q which is outside A. The set U on which we change ξ_t can contain arbitrary parts of the adjacent simplices as long as two different sets U do not meet because ξ_t and $\mathfrak{F}$ can never be parallel in one simplex if d is small enough.

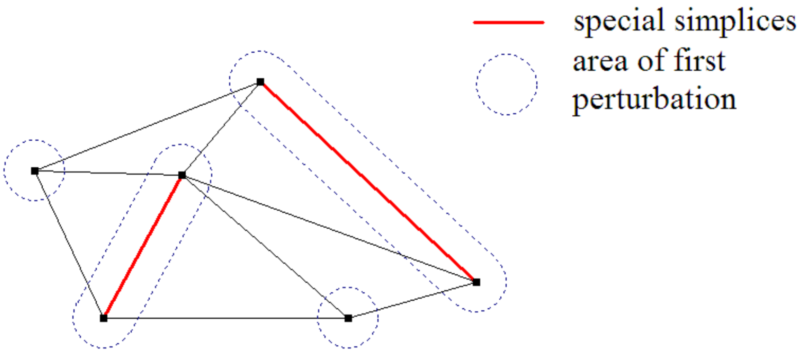

Figure 10: neighbourhoods of special simplices

Now we have to extend the perturbation to the non-special simplices. For this, we choose U as an $\varepsilon \cdot d$-neighbourhood of σ with

$$\varepsilon \cdot d < \frac{\delta}{4} \sin \frac{\alpha}{8} \iff \varepsilon < \frac{1}{4}\frac{\delta}{d} \sin \frac{\alpha}{8}$$

if σ is 1-dimensional and $\varepsilon < \frac{1}{8}\frac{\delta}{d}\sin^2\frac{\alpha}{8}$ in the 2-dimensional case (Notice at first, that ε does only depend on $\frac{\delta}{d}$ and α as we stated in condition 3). To prove that these non-special simplices fulfil the conditions of the above theorem (especially the condition on π_1), we look at a one-dimensional simplex σ. We will prove the π_1-condition for every leaf V of $\mathfrak{F}$; this way we reduce the problem to two dimensions. Around both vertices of σ there are neighbourhoods in which our contact structure is already defined and should not be changed. These are called A in the above theorem. Since we can choose their shape, they should get a triangle shape in V away from the endpoints of σ; near the endpoints A should be as wide as the neighbourhood, so that we can combine it with other simplices. The foliation on a leaf of $\mathfrak{F}$ then looks like this:

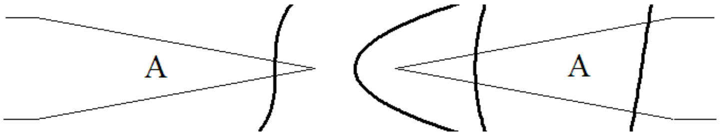

Figure 11: leaf of $\mathfrak{F}$ with foliation

The smallest angle of the triangle is chosen to be $\alpha/8$. The two triangles do not meet because of the condition on the width of U. The condition on π_1 will now be fulfilled exactly if every 1-dimensional leaf meets A at most once. We know that $\gamma(\sigma, \xi_t(x)) > \alpha/8$ is fulfilled in U, so the angle of every leaf to σ is greater than $\alpha/8$ at every point. Because of the shape of both parts of A, no leaf can enter one of these parts again after it has left it. A leaf cannot connect the left and the right part of A because of the condition on the width (the length of σ is greater than δ and the slope of the curve is greater than $\sin\frac{\alpha}{8}$). So every leaf hits A at most once. The 2-dimensional case is analogue.

Since the increase of the norm is controlled in every step, we can find a constant so that the norm of the resulting family of distributions is less than CN. By making d smaller, we now get ξ'_t $\mathcal{C}^0$-close to ξ_t. This way we also guarantee that the estimats (2) and (1) are still satisfied after every step. So we have found a family of plane distributions fulfilling the conditions we wanted. ☐

8 Curvature and Foliations

With the following two lemmata we will construct the simple foliations we described above by use of normal curvature.

Lemma 8.1. *Let σ be a 3-simplex of diameter d. Then for any $\lambda > 0$ there exists an embedded ball $B \subset \sigma$ such that its boundary ∂B is contained in*

λ-neighbourhood of $\partial\sigma$ and the normal curvatures of ∂B are everywhere $\geq 8\lambda/(4\lambda^2 + d^2)$.

Proof. We will use four spheres near the four faces of our simplex and smooth them at the intersection points. The 2-dimensional analogon looks like this:

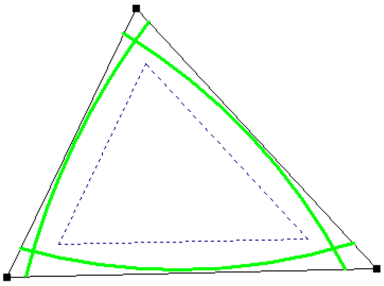

Figure 12: ball in simplex

Now we calculate the radius and centre of the spheres we use. We put the centre M of the sphere onto a line perpendicular to one face which runs through the centre F of that face. Let r be the distance between F and M which should also be the radius of the sphere. Let P be a point on the circle in which the sphere meets the boundary of the λ-neighbourhood and let α be $\angle PMF$.

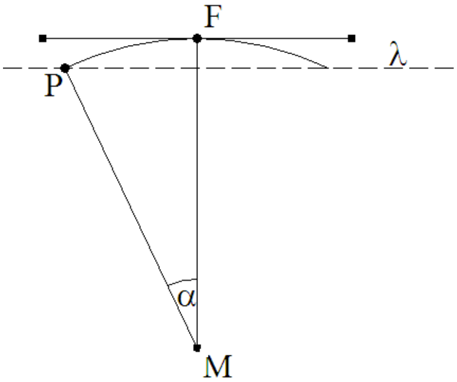

Figure 13: ball in simplex

So we have

$$(1 - \cos\alpha)r = \lambda$$

and want to have

$$\sin\alpha \cdot r \geq \frac{d}{2}$$
$$\frac{1}{r} \geq \frac{8\lambda}{4\lambda^2 + d^2}$$

By making the second inequality to an equation and using it as a definition for r, we get for $\cos\alpha$:

$$\begin{aligned} 1-\cos\alpha &= \frac{8\lambda^2}{4\lambda^2+d^2} \\ \cos\alpha &= \frac{d^2-4\lambda^2}{d^2+4\lambda^2} \end{aligned}$$

Now we have:

$$\begin{aligned} \sin\alpha &= \sqrt{1-\cos^2\alpha} \\ &= \sqrt{1-\left(\frac{d^2-4\lambda^2}{d^2+4\lambda^2}\right)^2} \\ &= \sqrt{\frac{16d^2\lambda^2}{(d^2+4\lambda^2)^2}} \\ &= \frac{4d\lambda}{d^2+4\lambda^2} \end{aligned}$$

By insertion we see that the first inequality now holds as well. So we have found sensible spheres. If we smooth them at intersection points we will get an even larger curvature there. ☐

Lemma 8.2. *Let $S \subset \mathbb{R}^3$ be an embedded 2-sphere with all normal curvatures $\geq K > 0$ and let ξ be a contact structure near S with $||\xi|| < K/2$. Then ξ is almost horizontal near S.*

Proof. The proof is based on the Poincaré-Bendixson theorem. Because of the non-integrability condition of contact structures, we have finitely many critical points. Near each critical point we get embedded circles on which the angle of the tangent plane to the tangent plane at the critical point is constant. We take a critical point C and assume WLOG that the orientations of ξ and T_CM at this point coincide (otherwise add a minus sign to one normal vector to get the same result). We will call vectors orthogonal to this circles in the direction of the critical point *vectors pointing inwards.*

Assertion: The oriented intersection of ξ with the tangent plane is never in a small neighbourhood of an inward pointing vector.

The intersection vector of two planes E_1 and E_2 can be written as the normal vector to the plane spanned by the normal vectors n_1 and n_2 of E_1 and E_2 respectively. Call the normal vector at C to T_CM by n. Consider a point P. The angle of the normal vector to T_PM to n is more than twice as big as the angle of the normal vector of $\xi(P)$ to n because of the curvature condition. So the plane spanned by both of them cannot be tangent to the circle of constant angle or near to it.

This implies that the index of all critical points is one, since a vector cannot turn around in the opposite direction or twice without becoming the inward pointing vector somewhere. As we work on S^2, we have exactly two critical points.

Following Poincaré-Bendixson, limit cycles can be circles or connect critical points (including one with itself). We will now exlude the case that a critical point is connected with itself. For this we will assume that our two critical points lie opposite each other on the sphere. Other cases can be treated similarly, but will be more confusing; the important fact is that we have two range of angles which cannot occur for tangent vectors, namely the inward pointing vectors for both critical points.

Consider a curve γ connecting a critical point C with itself. Assume that it has finite length in one of the real directions. Since the tagent vector to the curve at C is inward or outward pointing, the tangent vectors of γ near C have to approach this vector, and so have to come arbitrary close to being inward or outward poiting which is not possible (outward pointing vectors are inward pointing for the other critical point). So γ must have infinite length in both directions.

So let $\gamma : \mathbb{R} \to S$ be a curve with $\lim_{s\to\infty} = C = \lim_{s\to-\infty}$. Let d be the distance from C measured orthogonal to the circles of constant angle, and T be an arbitrary such circle. Take a disc neighbourhood (with distance function d) of C which is left by γ. Then the part of γ outside the neighbourhood is compact and contains a point P_0 so that $d(P)$ is maximal for points $P \in \gamma$. Let π be the projection of γ onto T. This projection has a lower bound for the norm of its derivative since otherwise the tangent vectors to γ would approach the inward or outward pointing vectors. So $\pi(\gamma(x))$ runs around the circle T infinitely often as x runs through the reals. Let P_1 be last point before and P_2 be the next point after P_0 for which we have $\pi(P) = \pi(P_1) = \pi(P_2)$. Since the derivative of π does not vanish, γ can be written as function of T in the parts from P_1 to P_0 and from P_0 to P_2. Let $h_1 : T \to \mathbb{R}$ be the distance of γ from C between P_1 and P_0, and h_2 be the same between P_0 and P_2. $h_1(x) - h_2(x)$ is negative near the beginning and positive near the end, since P_0 is the point with maximal distance. So there is $x_0 \in T$ with $h_1(x_0) = h_2(x_0)$. So there are two points on γ with the same projection π and the same distance, so they are equal. This is a contradiction. So there cannot be a curve which connects C with itself.

So all limit cycles are circles or there is a limit cycle which connects the two critical point, so that we have no circle leaves at all.

To show that all circle leaves are isolated, we can use Giroux's definition of *dividing set* on convex surfaces (see [6], p.11 or the originial paper "Convexité en topologie de contact" by Giroux). S is convex in this sense, since it is a $\mathcal{C}^\infty$ closed embedded surface. We take $S' \subset S$ with infinitely many circles and no critical points. We get by convexity a 1-dimensional manifold Γ on S', called the dividing set, with the properties:

- we have a unit vector field X on $S' \backslash \Gamma$ tangent to the leaves with everywhere positive divergence.
- X is transverse to Γ and points outward everywhere at the boundary

Every leaf that leaves $S'\backslash\Gamma$ somewhere must enter it somewhere else, since we have no singularities. Since all vectors at the boundary are outward pointing, we get two exactly opposite vectors when running from different directions to one point in Γ. By Gauß integral theorem

$$\int_V \operatorname{div} \vec{F}\, dV = \oint_{\partial V} \vec{F} \cdot \vec{n}\, dS,$$

we get:

$$\int_{S'\backslash\Gamma} \operatorname{div} X \; dS'\backslash\Gamma = 0.$$

This is not possible by the divergence condition.

Since a circle leaf has a critical point inside and outside, every limit cycle has to seperate the critical points. From Poincarè-Bendixson we know that every other leaf comes from or converges to limit cycles or critical points.

We can find the transversal connecting both poles by running orthogonal to the circles of constant angle from S to N, since these inward pointing vectors can never be equal to a tangent vector.

□

9 Proof of Classification Theorem

We will now prove the more general version of our classification theorem. First of all we state a lemma which is easy to prove if we look at the above section. The hypotheses will be the same as in the general classification theorem.

Lemma 9.1. *Let M be a compact 3-manifold and let A, $A \subset M$, be a closed subset such that $M\backslash A$ is connected. Let K be a compact space and L, $L \subset K$, a closed subspace. Let ξ_t, $t \in K$, be a family of plane distributions which are contact everywhere for $t \in L$ and are contact near A for $t \in K$. Suppose there exists an embedded 2-disc $\Delta \subset M\backslash A$ such that ξ_t is contact near Δ and (Δ, ξ_t) is equivalent to the standard overtwisted disc (Δ, ζ_0) for all $t \in K$.*

Then there exist disjoint 3-balls $B_1, \ldots, B_N \subset M\backslash(A \cup \Delta)$ and a family of distributions $\tilde{\xi}_t$, $t \in K$, on M such that

1. *$\tilde{\xi}_t$ coincides with ξ_t on $A \cup \Delta$ for $t \in K$ and everywhere for $t \in L$;*
2. *$\tilde{\xi}_t$, $t \in K$, is contact on $M\backslash \operatorname{Int} B_i$;*
3. *$\tilde{\xi}_t$, $t \in K$, is almost horizontal near ∂B_i, $i = 1, \ldots, N$;*
4. *$\tilde{\xi}_t$ is $\mathcal{C}^0$-close to ξ_t, $t \in K$.*
5. *$(B_i, \tilde{\xi}_t)$ for $t \in L$ and $i = 1, \ldots, N$ is isomorphic to a convex ball in $(\mathbb{R}^3, \zeta_0)$.*

Proof. Because this is an extension problem it is enough to consider the case when M is a compact domain in $\mathbb{R}^3$. Now we can apply theorem 7.5 and find a family $\tilde{\xi}_t$, $t \in K$, defined on a general simplicial complex P containing M with the following properties:

1. $\tilde{\xi}_t$ coincides with ξ_t on a subcomplex $A \cup \Delta \subset Q$ for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. This is immediate from 7.5.

2. there exists an $\varepsilon > 0$ which does not depend on $d = d(P)$ and such that $\tilde{\xi}_t$, $t \in K$, is contact in $\varepsilon \cdot d$-neighbourhood of the 2-skeleton of P. This is also directly given by 7.5.

3. $$||\tilde{\xi}_t|| < \frac{4\varepsilon}{d(4\varepsilon^2 + 1)}, t \in K,$$

   which we get by decreasing d.

Let $\sigma_1, \dots, \sigma_N$ be the sequence of all 3-simplices of $P \backslash Q$. By lemma 8.1 there exists a ball $B_i \subset \sigma_i$ for each $i = 1, \dots, N$ whose boundary ∂B_i is contained in the εd-neighbourhood of $\partial \sigma$ and has all normal curvatures greater than $\frac{8\varepsilon}{d(4\varepsilon^2+1)}$. So we can apply lemma 8.2 and see that $\tilde{\xi}_t$, $t \in K$ is almost horizontal near ∂B_i for $i = 1, \dots, N$. Condition 5 is a general property of contact structures. □

Now we want to connect all the balls:

Lemma 9.2. *Let* $\tilde{\xi}_t$ *be the foliation constructed above. Denote by* $\mathfrak{F}_t^i$ *the foliation* $(\partial B_i)_{\tilde{\xi}_t}$ *induced by* $\tilde{\xi}_t$ *on* ∂B_i *(*$t \in K$, $I = 1, \dots, N$*). Let* $\mathfrak{H}$ *be the foliation on the boundary of a small neighbourhood of the standard overtwisted disc* Δ *and let* B *denote the 3-ball. Then there exists a family of embeddings* $h_t : B \to M \backslash A$, $t \in K$, *such that for all* $t \in K$, $\cup_{i=1}^N B_i \cup \Delta \subset h_t(B)$ *and the foliation* $G_t = (h_t(\partial B))_{\tilde{\xi}_t}$ *induced by* $\tilde{\xi}_t$ *on the sphere* $h_t(\partial B)$ *is homeomorphic to the connected sum* $\mathfrak{H} \# (\#_{i=1}^N \mathfrak{F}_t^i)$*; for* $t \in L$ *the embedding* h_t *defines a contact isomorphism of* (B, ζ_1) *and* $(h_t(B), \xi_t)$*.*

Proof. Let $M_1 = M \backslash \left(\bigcup_1^N \operatorname{Int} B_i \cup A \right)$. Let B_0, $\Delta \subset B_0$, be the ball in M_1 such that the foliation $(\partial B_0)_{\tilde{\xi}_t}$ is homeomorphic to $\mathfrak{H}$. Let $M_1' = M_1 \backslash B_0$. Orient $\tilde{\xi}_t$ near $\cup_{i=0}^N \partial B_i$.

Let us connect the north pole of $\mathfrak{H}$ with the south pole of $\mathfrak{F}_t^1$ by an embedded curve $\mathfrak{l}_t^0 \subset M_1'$ and then consequently for $i = 1, \dots, N-1$ connect the north pole of $\mathfrak{F}_t^i$ with the south pole of $\mathfrak{F}_t^{i+1}$ by embedded disjoint curves $\mathfrak{l}_t^i \subset M_1'$.

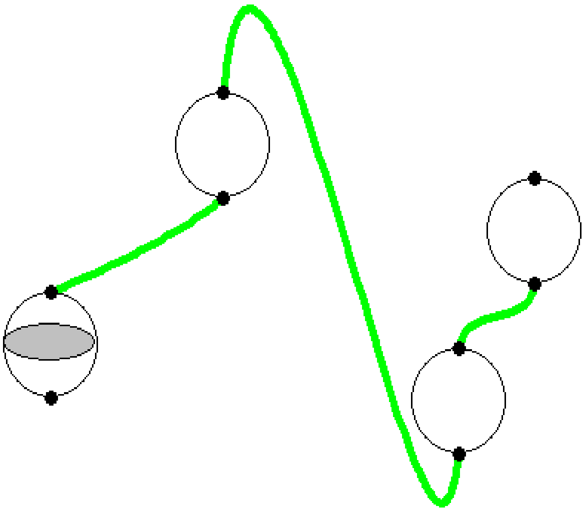

Figure 14: connection of balls by transversals

We should adjust the orientation so that $\tilde{\xi}_t$ is oriented on $\cup_1^N \partial B_t \cup \mathfrak{l}_t^i$. With the use of theorem 6.3 we can make all the $\mathfrak{l}_t^i$ transverse to $\tilde{\xi}_t$. Around a transverse curve we can find a cylinder with the contact structure $dz + xdy + ydx$, where z is the direction of $\mathfrak{l}$ (see e.g. [3]).

Let us consider B_i and B_{i+1}. We now want to connect the balls to one ball by a small cylinder near the transversals. The cylinder should be chosen parallel to and inside the cylinder with the above contact structure and thin enough so that it meets B_i and B_{i+1} above (resp. below) the nearest limit cycle to the critical point. If we smooth the connection we get one ball with foliation on the boundary. The foliation on the cylinder does not contain circles: Every circle must have a point with zero derivative in z-direction, but at such a point the tangent vector is no zero of the contact form. So the constructed foliatiated surface has a foliation of the topological type of the connected foliation. Following this procedure, we can connect all balls to one.

For $t \in L$ we connect one ball with the standard overtwisted contact structure with several balls with the standard contact structure, so we get a foliation with just two circles which is the same as for a neighbourhood of the standard overtwisted disc. ☐

Remark 9.3. Because the foliation $\#_{i=1}^N \mathfrak{F}_t^i$, $t \in K$, is almost horizontal (all the balls have an almost horizontal foliation which can be extended onto the cylinders), it can be defined up to a homeomorphism by the family of holonomy diffeomorphisms $\psi_t : I \to I$, $t \in K$. Since circles in the foliation correspond to fixed points in ψ_t, we have no fixed points for $t \in L$, and isolated ones for $t \in K \backslash L$.

Definition 9.4. Let $\beta : [0,1] \to (-\pi/4, \pi/4)$ be a function with isolated zeros and with $\beta(0), \beta(1) < 0$. Let γ_β be a curve in the plane with coordinates (ρ, z) as shown in Figure 15 and $S_\beta \subset \mathbb{R}^3$ be the surface of revolution of γ_β around the z-axis.

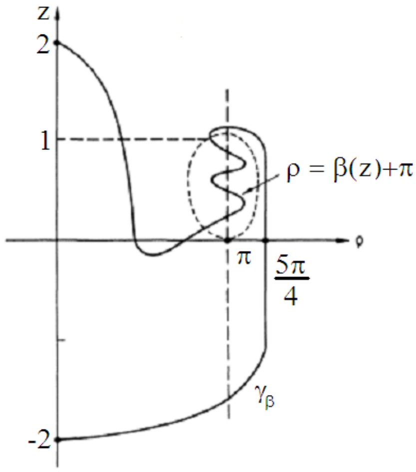

Figure 15: function γ_β (from [1] with different labelling)

Let ζ_1 be the standard overtwisted contact structure in $\mathbb{R}^3$.

Denote by $\mathfrak{H}_\beta$ the foliation $(S_\beta)_{\zeta_1}$ induced on S_β by ζ_1 (see Figure 16). If β has no zeros, then $\mathfrak{H}_\beta$ is homeomorphic to $\mathfrak{H}$.

Figure 16: surface of revolution (from [1])

Theorem 9.5. *Let $f_t : I \to I$, $t \in K$, be a family of diffeomorphisms with isolated fixed points inside, fixed at ∂I and satisfying the condition $f_t(x) > x$ near ∂I. Let $g_t(x) = \pi/4(f_t(x) - x)$ for $t \in K$, $x \in I$. Then the family $\mathfrak{H}_{g_t}$, $t \in K$ is topologically equivalent to the family $\mathfrak{H}\#\mathfrak{F}(f_t)$, $t \in K$.*

Proof. According to lemma 5.5 it is enough to verify that both families of foliations have the same family of diagrams. Since zeros of g_t are exactly fixed points of f_t and since γ_t crosses the π-cylinder exactly if g_t has a zero, it is easy to see that this is true. $\square$

Proof of the classification theorem. Using the lemmata 9.1 and 9.2 we know that there exists a family of distributions $\tilde{\xi}_t$, $t \in K$, and a family of embeddings $h_t : B \to M \backslash A$, $t \in K$, such that $\tilde{\xi}_t$ is $\mathcal{C}^0$-close to ξ_t, $t \in K$; $\tilde{\xi}_t$ coincides with ξ_t on A for $t \in K$ and coincides with ξ_t everywhere for $t \in L$; $\tilde{\xi}_t$ is contact outside of $h_t(B) \subset M \backslash A$, $t \in K$; and the foliation $(h_t(\partial B))_{\tilde{\xi}_t}$, $t \in K$, is homeomorphic to $\mathfrak{H}\#\mathfrak{F}(\psi_t)$ where $\psi_t : I \to I$ is a diffeomorphism with finite number of fixed points.

By theorem 9.5 the family $\mathfrak{H}\#\mathfrak{F}(\psi_t)$, $t \in K$, is topologically equivalent to the family $\mathfrak{H}_{\beta_t}$ for a family of functions $\beta_t : [0,1] \to (-\pi/4, \pi/4)$ with finite number of zeros. But the structure ζ_1 which induces the foliation $\mathfrak{H}_{\beta_t}$ on the sphere S_{β_t} is extended to the ball bounded by S_{β_t}.

Hence we conclude by theorem 5.5 that the family $\tilde{\xi}_t$, $t \in K$, of contact structures near $h_t(\partial B)$ which induces the family of foliations $(h_t(\partial B))_{\tilde{\xi}_t}$ are extendable to $h_t(\partial B)$ as a family ξ'_t of contact structures. Since a ball is contractible, we can connect ξ_t and ξ'_t by a homotopy.

Note that in view of lemma 9.2, $(h_t(B), \xi_t)$ for $t \in L$ is isomorphic to the standard overtwisted ball (B, ζ_1). But our construction provides the same property for $(h_t(B), \xi'_t)$, $t \in L$. Hence we can choose $\xi_t = \xi'_t$ for $t \in L$. $\square$

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