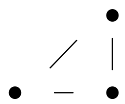
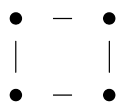
1. Let $\Gamma = (V, E)$ be a simple, connected, locally finite graph. The diameter of $\Gamma = (V, E)$ is defined by
$$\operatorname{diam} \Gamma = \sup_{x,y \in V} d(x, y).$$
   (a) Prove that Γ is finite if and only if $\operatorname{diam} \Gamma < \infty$.
   (b) Prove that Γ is a complete graph K_n if and only if $\operatorname{diam} \Gamma = 1$.
   (c) Prove that, for any vertex x and for any positive integer $n \leq \frac{1}{2} \operatorname{diam} \Gamma$, there is a vertex $y \in V$ such that $d(x, y) = n$.

2. A simple connected graph is called bipartite if it admits a coloring of its vertices into two colors, say, black and white, so that the vertices of the same color are not connected by an edge (for example, the set of fields of a chessboard is a bipartite graph). Prove that a simple connected graph is bipartite if and only if it contains no cycle C_n with odd n.

   Hint: Chose the color of a vertex x depending on the distance $d(x, x_0)$ where x_0 is a fixed vertex.

3. A graph (V_1, E_1) is said to be a *subgraph* of (V, E) if $V_1 \subset V$ and $E_1 \subset E$. Two graphs (V_1, E_1) and (V_2, E_2) are said to be *isomorphic* if there is a bijection $\varphi : V_1 \hookrightarrow V_2$ that preserves edges, that is, $(\varphi(x), \varphi(y)) \in E_2$ if and only if $(x, y) \in E_1$. Given two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma = (V, E)$, denote by $N(\Gamma, \Gamma_1)$ the number of distinct subgraphs of Γ that are isomorphic to Γ_1. For example, let K_n be a complete graph with n vertices and C_n be a cycle with n vertices. Then $N(\Gamma, K_1)$ is the number of vertices of graph Γ, $N(\Gamma, K_2)$ is the number of edges of Γ, $N(\Gamma, K_3)$ is the number of complete triangles  to be found in Γ, $N(\Gamma, C_4)$ is the number of 4-cycles  to be found in Γ.

   Evaluate $N(K_n, C_k)$ for all $n \geq k \geq 3$.

4. Let $K_{n,m}$ be a complete bipartite graph. Evaluate $N(K_{n,m}, C_k)$ for any $k \geq 3$.

5. A complete m-partite graph $K_{n_1,\ldots,n_m}$ is defined as follows. It has $n_1 + \ldots + n_m$ vertices that are split into m groups $V_1, \ldots, V_m$ such that $\# V_k = n_k$, and two vertices x, y are connected if and only if they belong to different groups V_k. Prove that if $n_1 = \ldots = n_m = n$ then the graph $K_{n,\ldots,n}$ is a Cayley graph.

6. Prove that the numbers $a_j = N(K_{n_1,\ldots,n_m}, K_j)$ satisfy the following identity:
$$(z - n_1)(z - n_2) \ldots (z - n_m) = z^n - a_1 z^{n-1} + a_2 z^{n-2} + \ldots + (-1)^m a_m$$
   for all complex z.

7. A subset S of a group G is called generating if any element $x \in G$ can be represented in the form
$$x = s_1 * s_2 * \ldots * s_n$$

for some positive integer n and with some $s_k \in S$. Prove that if S is a symmetric generating subset of G then the Cayley graph (G, S) is connected. (A graph (V, E) is called connected if, for any two vertices $x, y \in V$, there is a path in (V, E) connecting x and y.)

8. A graph (V, E) is called regular if $\deg(x)$ is the same for all $x \in V$. The following graphs are obviously regular: all Cayley graphs, cycles C_n, complete graphs K_n, complete multipartite graphs $K_{n,\dots,n}$, and their products.

   (a) List all connected regular graphs with at most 6 vertices. Show that every such graph is a Cayley graph. Show that every such graph belongs to one of the families $C_n, K_n, K_n \Box K_m, K_{n,..,n}$.

   (b) Give an example of a connected regular graph with 7 vertices that is non-Cayley and that does not belong to any family $C_n, K_n, K_n \Box K_m, K_{n,..,n}$.

9. Let P, Q be Markov kernels on a finite or countable set V. Consider a function $P \circ Q$ on $V \times V$ that is defined by

$$(P \circ Q)(x, y) = \sum_{z \in V} P(x, z) Q(z, y).$$

   (a) Prove that $P \circ Q$ is a Markov kernel.

   (b) Prove that if R is a Markov kernel then

$$(P \circ Q) \circ R = P \circ (Q \circ R).$$

   (c) Prove that if P and Q are reversible with respect to the same function $\mu(x)$ and $P \circ Q = Q \circ P$ then $P \circ Q$ is also reversible with respect to $\mu(x)$.

10. Let (V, μ) be a weighted simple graph with m vertices. Prove that $\operatorname{trace} \Delta_\mu = -m$, where Δ_μ is the Laplace operator of (V, μ) and $\operatorname{trace} \Delta_\mu$ is its trace.

    Hint: Write the Laplace operator as a matrix in some basis and then evaluate the trace as the sum of the diagonal elements.

11. * Let a simple graph Γ have m vertices and n edges.

    (a) Prove that if $n \geq \left[\frac{m^2}{4}\right] + 1$ then $N(\Gamma, K_3) \geq \left[\frac{m}{2}\right]$.

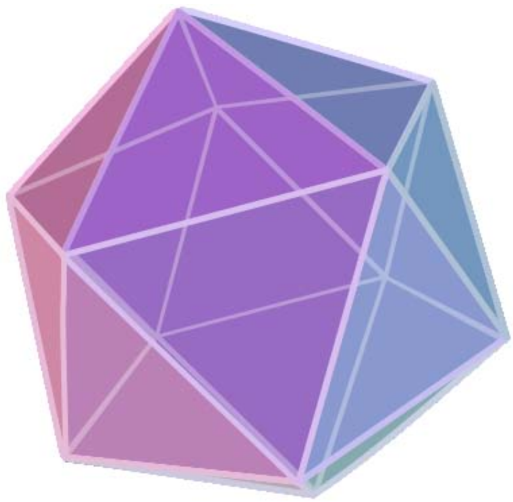
    (b) For any positive integer m, give and example of a graph Γ with $n = [\frac{m^2}{4}]$ edges such that $N(\Gamma, K_3) = 0$.

12. * Let A_4 be the group of even permutations of the set $\{1, 2, 3, 4\}$. Consider the cyclic permutations $a = (234)$ and $b = (123)$ as well as the product of two transpositions $c = (12)(34)$.

    (a) Verify that $a^3 = b^3 = c^2 = \mathrm{id}$ and $ba = a^2 b^2 = c$.

    (b) Prove that A_4 is isomorphic to the group of rotations of a regular tetrahedron in $\mathbb{R}^3$.

(c) Consider a symmetric set $S = \{a, a^{-1}, b, b^{-1}, c\}$ and prove that the Cayley graph (A_4, S) is isomorphic to the icosahedron graph:



13. Let $u(x) \geq 0$ be a function on a weighted graph (V, μ) such that $\Delta_\mu u + c(x) u^p \leq 0$ in a connected subset $\Omega \subset V$, for some function $c(x)$ in Ω and constant $p > 0$. Prove that if $u(x) = 0$ at some vertex $x \in \Omega$ then $u \equiv 0$ in Ω (*a strong minimum principle*).

14. Prove the following identities for arbitrary functions f, g on a weighted graph (V, μ):

    (a) $\nabla_{xy}(fg) = (\nabla_{xy} f) g + (\nabla_{xy} g) f + (\nabla_{xy} f)(\nabla_{xy} g)$.

    (b) $\Delta_\mu (fg) = (\Delta_\mu f) g + (\Delta_\mu g) f + \frac{1}{\mu(x)} \sum_{y \sim x} (\nabla_{xy} f)(\nabla_{xy} g) \mu_{xy}$.

15. Consider the equation $\Delta_\mu u = f$ on a finite connected weighted graph (V, μ). Here f is a given function whereas u is an unknown function.

    (a) Prove that if one solution u exists then all other solutions are $u + \text{const}$.

    (b) Prove that if a solution u exists then

$$\sum_{x \in V} f(x) \mu(x) = 0. \tag{1}$$

    (c) Prove that if (1) is satisfied the a solution u exists.

16. Let $\{X_n\}$ be a simple random walk on $\mathbb{Z}$, and set $v_n(x) = \mathbb{P}_0(X_n = x)$.

    (a) Prove that

$$v_n(x) = \begin{cases} \frac{1}{2^n} \binom{n}{\frac{x+n}{2}}, & x \equiv n \mod 2 \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

    where $\binom{n}{m}$ is the binomial coefficient that is defined by

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!} & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Prove that, for even n, $v_n(0) \sim \sqrt{\frac{2}{\pi n}}$ as $n \to \infty$. *Hint:* Use the Stirling formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

17. Let $\mathcal{F}$ be the space of real-valued functions on a finite weighted graph (V,μ), endowed by the inner product

$$(f,g) = \sum f(x)\, g(x)\, \mu(x).$$

Set $\|f\| = \sqrt{(f,f)}$. Let P be the Markov operator of (V,μ) and $\mathcal{L}$ be the positive definite Laplace operator of (V,μ).

(a) Prove that, for any $f \in \mathcal{F}$, $(Pf)^2 \le P(f^2)$

(b) Prove that, for any $f \in \mathcal{F}$, $\|Pf\| \le \|f\|$.

(c) Use (b) to show that $\operatorname{spec} P \in [-1,1]$. Conclude that $\operatorname{spec}\mathcal{L} \subset [0,2]$.
Remark: This gives an alternative proof of the fact that all the eigenvalues of $\mathcal{L}$ are contained in $[0,2]$.

18. Let (X,E_1) and (Y,E_2) be two finite connected graphs with more than one vertex. Prove that their product $(X,E_1)\,\square\,(Y,E_2)$ is bipartite if and only if both (X,E_1) and (Y,E_2) are bipartite.

19. Let P be the Markov kernel of a locally finite weighted graph (V,μ) and E be the corresponding set of edges.

(a) Fix a positive integer n, and consider two vertices $x, y \in V$. Prove that $P_n(x,y) > 0$ if and only if there is a path of length n in the graph (V,E) that connects x and y.

(b) Define a new set of edges E_n on V as follows: $xy \in E_n$ if $P_n(x,y) > 0$. Prove that if (V,E) bipartite then (V,E_2) is disconnected.

(c) Let (V,E) be finite, connected and non-bipartite. Prove that (V,E_n) is complete for some n.

20. * Let (V,μ) be a finite connected weighted graph. Assume that (V,μ) is bipartite, and let V^+, V^- be a partition of V into two disjoint subsets such that $x \sim y$ implies that x, y belong to different subsets V^+, V^-.

(a) For any function f on V, consider the function $\widetilde{f}$ on V that takes two values as follows:

$$\widetilde{f}(x) = \frac{2}{\mu(V)} \begin{cases} \sum_{y \in V^+} f(y)\,\mu(y), & x \in V^+, \\ \sum_{y \in V^-} f(y)\,\mu(y), & x \in V^-. \end{cases}$$

Prove that if n is even and $n \to \infty$ then $P^n f(x) \to \widetilde{f}(x)$ for all $x \in V$.

(b) Consider the distribution $v_n(x) = \mathbb{P}_{x_0}(X_n = x)$ of the random walk on (V,μ). Prove that if $x_0 \in V^+$ and n is even then as $n \to \infty$

$$v_n(x) \to \begin{cases} \frac{2\mu(x)}{\mu(V)}, & x \in V^+, \\ 0, & x \in V^-. \end{cases}$$

21. Prove that the positive definite Laplace operator $\mathcal{L}$ on a complete bipartite graph $K_{n,m}$ (where $n + m > 2$) with simple weight has the following eigenvalues: $0, 1, 2$. What are their multiplicities?

22. Prove that the positive definite Laplace operator $\mathcal{L}$ on a complete m-partite graph $K\underbrace{n, n, ..., n}_{m}$ (where $m \geq 2$ and $n \geq 2$) with simple weight has the following eigenvalues: $0, 1, \frac{m}{m-1}$. What are their multiplicities?

23. * (*The Dirichlet principle*) Let Ω be a finite set of vertices on a connected weighted graph (V, μ) such that Ω^c is non-empty. Consider the Dirichlet problem

$$\begin{cases} \Delta_\mu u(x) = 0 & \text{for all } x \in \Omega, \\ u(x) = g(x) & \text{for all } x \in \Omega^c, \end{cases} \tag{3}$$

where g is a given function on Ω^c.

(a) Prove that a solution u of the Dirichlet problem (3) has the smallest value of the Dirichlet integral

$$D(u) = \frac{1}{2} \sum_{x,y \in \overline{\Omega}} (\nabla_{xy} u)^2 \mu_{xy},$$

among all other functions u that satisfy the same boundary condition $u = g$ in Ω^c. Here $\overline{\Omega}$ is the union of Ω with all its neighbors.

(b) Prove that if u minimizes the Dirichlet integral among all functions with the boundary condition $u = g$ in Ω^c then u is solves (3).

(c) Prove that there exists a function u that minimizes the Dirichlet integral among all functions with the boundary condition $u = g$ in Ω^c. *Remark:* This provides an alternative proof of the existence of solution of (3).

24. Prove that if (X, E_1) and (Y, E_2) are two connected graph then the graph $(V, E) = (X, E_1) \square (Y, E_2)$ is also connected.

25. Let (V, μ) be a finite connected weighted graph without loops. Let $\lambda_0 = 0 < \lambda_1 \leq ... \leq \lambda_{N-1}$ be the eigenvalues of the Laplace operator $\mathcal{L}$ on (V, μ).

(a) Assume that, for some positive integer k, there are $k+1$ functions $f_1, f_2, ..f_{k+1}$ on V such that

(*i*) their supports $A_i = \{x \in V : f_i(x) \neq 0\}$ are disjoint and not connected, that is, if $x \in A_i$ and $y \in A_j$ with $i \neq j$ then $x \neq y$ and $x \not\sim y$.

(*ii*) $\mathcal{R}(f_i) \leq a$ for some real a and for all $i = 1, 2, ..., k+1$, where $\mathcal{R}(f)$ is the Rayleigh quotient of f.

Prove that $\lambda_k \leq a$.

(b) Let D be the diameter of the graph (V, μ), that is,

$$D = \max_{x,y \in V} d(x, y).$$

Prove that, for any $k \leq [D/2]$, we have $\lambda_k \leq 1$.

26. * Let (V, E) be a connected locally finite infinite graph without loops. A finite or infinite sequence $\{x_k\}$ of vertices on V is called a *geodesic* if $d(x_k, x_n) = |k - n|$ for all indices k, n. Prove that there is an infinite geodesic starting at any given vertex $x \in V$.

27. Evaluate the eigenvalues and eigenfunctions of the Markov operator on a *path graph* (V, E) with simple weight, that is, $V = \{0, 1, ..., N-1\}$ and the edges are defined by
$$0 \sim 1 \sim ... \sim N-1.$$

28. Let (V, μ) be a finite connected weighted graph with $N > 1$ vertices. Let P be the Markov operator on (V, μ), $\{v_k\}_{k=0}^{N-1}$ be an orthonormal basis of eigenfunctions of P with eigenvalues α_k, where $1 = \alpha_0 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{N-1}$. Fix a point $x_0 \in V$ and set $f = \mathbf{1}_{\{x_0\}}$.

    (a) Assume that there is a constant c such that $|v_k(x)| \leq c$ for all $x \in V$ and $k = 1, ..., N-1$. Prove that
    $$\left| P^n f(x) - \frac{\mu(x_0)}{\mu(V)} \right| \leq c^2 \mu(x_0) \sum_{k=1}^{n} |\alpha_k|^n$$
    for all $x \in V$ and positive integers n.

    (b) * Prove that if (V, μ) is a cycle graph $C_N = \mathbb{Z}_N$ with an odd N and a simple weight μ then
    $$\left| P^n f(x) - \frac{1}{N} \right| \leq \frac{4}{N} \frac{1}{e^{\frac{4n}{N^2}} - 1}$$
    for all $x \in V$ and positive integers n. Conclude that the mixing time T admits the estimate $T \approx N^2$.

    Hint: Use the explicit eigenvalues and eigenfunctions of $\mathbb{Z}_N$ and the inequality $0 \leq \cos z \leq e^{-\frac{z^2}{2}}$ for $z \in (0, \pi/2)$.

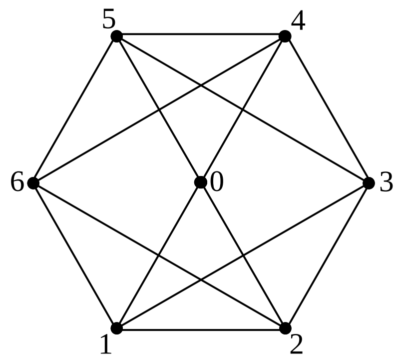
29. Let (V, E) be a finite connected regular graph of degree k. Let $a, b \in V$ be two distinct vertices of V such that $x \sim a$ implies $x \sim b$. Prove that the following function on V
$$f(x) = \begin{cases} 1, & x = a \\ -1, & x = b \\ 0, & \text{otherwise} \end{cases}$$
is an eigenfunction of the Laplace operator $\mathcal{L}$ on (V, E) (with a simple weight). What is its eigenvalue?

30. Let (V, μ) be a finite connected weighted graph and i be a weight preserving involution of (V, μ), that is, a non-identical mapping $i : V \to V$ such that $i^2 = \text{id}$ and $\mu_{i(x)i(y)} = \mu_{xy}$ for all $x, y \in V$.

    (a) Prove that there exists a non-constant eigenfunction $f(x)$ of the Laplace operator $\mathcal{L}$ on (V, μ) such that $f \circ i = -f$.

    (b) Prove that if there exist vertices $x_1, x_2 \in V$ such that the four vertices $x_1, x_2, i(x_1), i(x_2)$ are all distinct then there exists a non-constant eigenfunction $f(x)$ of the Laplace operator $\mathcal{L}$ on (V, μ) such that $f \circ i = f$.

31. * Evaluate the eigenvalues of the Laplace operator for the following graph with simple weight:



Hint: Use Exercises 29 and 30 to build various eigenfunctions of the Laplace operator.

32. * Let (V, μ) be a finite connected weighted graph. Prove that if the diameter of a graph is $D \geq 1$ then there are at least $D + 1$ distinct eigenvalues of the Markov operator P.

33. Let (V, μ) be a finite connected weighted graph. Prove that, for any subset $\Omega \subset V$,

$$\mu(\partial\Omega) \geq \lambda_1 \frac{\mu(\Omega)\,\mu(\Omega^c)}{\mu(V)},$$

where λ_1 is the smallest positive eigenvalue of the Laplace operator on (V, μ).

34. Give an example of a non-complete connected weighted graph (V, μ) with $N = 3$ vertices and disjoint non-empty subsets X, Y of V such that

$$d(X, Y) = 1 + \frac{l(X, Y)}{\ln \frac{\lambda_2 + \lambda_1}{\lambda 2 - \lambda_1}},$$

where $l(X, Y) = \ln \sqrt{\frac{\mu(X^c)\mu(Y^c)}{\mu(X)\mu(Y)}}$.

35. Let (V, μ) be a finite connected weighted graph with $N > 1$ vertices. Fix a positive integer r and define the *expansion factor* F_r of the graph by

$$F_r = \inf_{X \subset V} \frac{\mu(X_r \setminus X)\,\mu(V)}{\mu(X)\,\mu(X^c)},$$

where $X_r = \{x \in V : d(x, X) \leq r\}$. Prove that

$$F_r \geq 1 - \left(\frac{\lambda_{N-1} - \lambda_1}{\lambda_{N-1} + \lambda_1}\right)^{2r},$$

where λ_1 and λ_{N-1} are the eigenvalues of the Laplace operator on (V, μ).

36. Fix a positive integer N and consider the following subset Ω of $\mathbb{Z}^2$:

$$\Omega = \{(j, 0) : j = 1, 2, ..., N\}.$$

Evaluate all the eigenvalues and eigenfunctions of the Dirichlet Laplace operator $\mathcal{L}_\Omega$.

37. Let Ω be a finite non-empty subset of an infinite, locally finite, connected, weighted graph (V,μ). Prove that if $\frac{(\mathcal{L}_\Omega f,f)}{(f,f)} = \lambda_1(\Omega)$ for some non-zero function $f \in \mathcal{F}_\Omega$ then f is an eigenfunction of $\mathcal{L}_\Omega$ with the eigenvalue $\lambda_1(\Omega)$.

38. Let (V,E) be an infinite connected graph that is m-regular, and let μ be a simple weight on (V,E). Let Ω be a finite non-empty subset of V such that every vertex of Ω has at most k neighbors in Ω where $2 \le k < m$ (for example, if Ω is a path or a cycle then $k=2$). Prove that

$$h(\Omega) \ge \frac{m-k}{m} \quad \text{and} \quad \lambda_1(\Omega) \ge \frac{1}{2}\left(\frac{m-k}{m}\right)^2.$$

39. (a) Let (V,μ) be an infinite, locally finite, connected, weighted graph. Let Ω be a non-empty finite subset of V and let r be a positive integer. Set $\Omega_r = U_r(\Omega)$ and prove that

$$\lambda_1(\Omega_r) \le \frac{\mu(\Omega_{r+1})}{r^2 \mu(\Omega)}.$$

(b) Let $B_r = \{x \in \mathbb{Z}^m : d(x,0) \le r\}$ be the ball of radius r in $\mathbb{Z}^m$. Prove that

$$\lambda_1(B_r) \le \frac{C}{r^2}$$

and

$$\lambda_1(B_r) \le C' \mu(B_r)^{-2/m},$$

where C and C' are constants depending only on m.

In all questions, (V,μ) is an infinite, locally finite, connected weighted graph, and Ω is a finite non-empty subset of V.

40. Consider the Dirichlet problem $\mathcal{L}_\Omega u = f$ where f and u are functions from $\mathcal{F}_\Omega$. Prove the following inequalities:

(a) $\|u\| \ge \frac{1}{2}\|f\|$.

(b) $\|u\| \le \frac{1}{\lambda_1(\Omega)}\|f\|$.

(c) Let $\Omega_1 = U_1(\Omega)$. Then

$$\frac{1}{2}\sum_{x,y\in\Omega_1} (\nabla_{xy} u)^2 \mu_{xy} \le \frac{1}{\lambda_1(\Omega)}\|f\|^2.$$

41. Let u be a subharmonic function on V, that is, $\mathcal{L}u \le 0$ on V. Prove that if, for some point $x \in V$,

$$u(x) = \sup u,$$

then $u \equiv \text{const}$. In other words, a subharmonic function on V cannot attain its supremum unless it is a constant *(a strong maximum principle).*

42. In this and the next two questions, assume that

$$c := \inf_{x,y \in V: x\sim y} \mu_{xy} > 0.$$

Let x_0 be a vertex from Ω. Prove that, for any function $f \in \mathcal{F}_\Omega$ such that $f(x_0) = 1$, the following inequality holds:

$$(\mathcal{L}_\Omega f, f) \geq \frac{c}{d(x_0, \Omega^c)}.$$

Hint: Consider a path $\{x_k\}_{k=0}^n$ connecting x_0 to the nearest point from Ω^c and use the sum $\sum_{k=1}^n (f(x_{k-1}) - f(x_k))^2 \mu_{x_{k-1}x_k}$.

43. Let R be *inradius* of Ω, that is,

$$R = \max\{r \geq 0 : B_r(x) \subset \Omega \text{ for some } x \in \Omega\},$$

where $B_r(x) = \{y \in V : d(x,y) \leq r\}$. Prove that

$$\lambda_1(\Omega) \geq \frac{c}{(R+1)\mu(\Omega)}.$$

Hint: Use Exercise 42.

44. * Assume that, for all $x \in V$ and all $r \geq 0$,

$$\mu(B_r(x)) \geq ar^m,$$

with some positive constants a, m. Prove that (V, μ) satisfies the Faber-Krahn inequality with the function

$$\Lambda(s) = bs^{-\frac{m+1}{m}},$$

where $b = b(a, c, m) > 0$. *Hint*: Use Exercise 43.

45. * Let Ω be connected and f be an eigenfunction of $\mathcal{L}_\Omega$ with the eigenvalue $\lambda_1(\Omega)$.

    (a) Prove that if $f \geq 0$ then $f > 0$ in Ω.

    (b) Prove that f_+ and f_- are also the eigenfunctions of $\lambda_1(\Omega)$, provided they do not vanish identically. Here $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$ so that $f = f_+ - f_-$. *Hint*: Use Exercise 37.

    (c) Prove that either $f > 0$ in Ω or $f < 0$ in Ω. *Hint*: Assume the contrary and use (b).

    (d) Prove that $\lambda_1(\Omega)$ is a simple eigenvalue. *Hint*: Assuming that there exist two linearly independent eigenfunctions, consider their linear combination that vanishes at some vertex, and use (c).

In all problems (V, μ) is a locally finite, connected, infinite weighted graph, P is the corresponding Markov kernel, and $p_n(x, y)$ is the heat kernel.

46. Let f be a function with finite support on (V, μ). Set $u_n(x) = P^n f(x)$.

    (a) Prove that $\sup_x |u_n(x)|$ is a decreasing function of n.

    (b) Prove that $\|u_n\|$ is a decreasing function of n.

    (c) Prove that the heat kernel $p_{2n}(x, x)$ is a decreasing function of n, for any fixed vertex x.

47. Assume that $\mu(x) \geq 1$ for all $x \in V$ and that the heat kernel on (V, μ) satisfies the following estimate: $p_n(x, x) \leq Cn^{-\alpha}$ for all $x \in V$ and all positive integers n, where C and α are positive constants. Prove that, for any $0 < \varepsilon < \alpha$, for all $x, y \in V$ and all positive integers n,

$$p_n(x, y) \leq \frac{C'}{n^{\alpha-\varepsilon}} \exp\left(-c\frac{d^2(x, y)}{n}\right)$$

where C' and c are some positive constants.

Hint: Prove first that $p_n(x, y) \leq \operatorname{const} n^{-\alpha}$ for all x, y and n, and then combine this estimate with the estimate of Carne-Varopoulos.

48. Prove that if (V, μ) is a Cayley graph with the exponential volume growth then the heat kernel of (V, μ) admits the following estimate

$$p_n(x, y) \leq C \exp\left(-\frac{d^2(x, y)}{4n} - cn^{1/3}\right),$$

for all $x, y \in V$ and positive integers n, and some constants $C, c > 0$.

49. Assume that there exists a constant $p_0 > 0$ such that

$$P(x, y) \geq p_0 \text{ for all } x \sim y.$$

(a) Prove first that $\deg(x) \leq 1/p_0$ for any vertex $x \in V$.

(b) Prove that, for all $x, y \in V$,

$$\mu(x) \geq p_0^{d(x,y)} \mu(y)$$

(c) Prove that any ball $B(x, r)$ contains at most C^r vertices where $C = C(p_0)$.

(d) Prove that, for any finite set $A \subset V$ and for any positive integer r,

$$\mu(U_r(A)) \leq K^r \mu(A)$$

where $K = K(p_0)$.

50. Assume that $\mu(x) \geq 1$ for all $x \in V$, $\mu(B(X, r)) \leq Cr^\alpha$ for all $x \in V$ and positive integers r, and that

$$p_n(x, y) \leq \frac{C}{n^{\alpha/2}} \exp\left(-c\frac{d^2(x, y)}{n}\right)$$

for all $x, y \in V$ and positive integers n, where C, c, α are positive constants (for example, all these hypotheses hold on $\mathbb{Z}^m$ with $m = \alpha$ or, more generally, on any Cayley graph of polynomial volume growth). Prove that, for all $x \in V$ and for all positive even integers n,

$$p_n(x, x) \geq \frac{c'}{n^{\alpha/2}}$$

with some positive constant c'.

Hint: Use the computation from the proof of a theorem about lower estimate of the heat kernel under the volume growth condition.

51. For any self-adjoint operator T, denote by $\alpha_{\max}(T)$ its maximal eigenvalue. For any non-empty finite set $U \subset V$, denote by P_U the restriction of the Markov operator P to functions on U, that is,

$$P_U f(x) = \sum_{y \in U} P(x,y) f(y).$$

Prove that

$$\alpha_{\max}\left((P_U)^2\right) \leq \alpha_{\max}(P_U).$$