Prof. A. Grigoryan — Analysis on Manifolds — SS 2024

Blatt 3. Abgabe bis 10.05.2024

16. Let M be a smooth manifold of dimension n, $F \in C^\infty(M)$ and S be a non-singular null set of F, that is,

$$S = \{x \in M : F(x) = 0\} \text{ and } dF \neq 0 \text{ on } S.$$

Consequently, S is a submanifold of M of dimension $n-1$.

(a) Prove that, for any $x_0 \in S$, the tangent space $T_{x_0}S$ is determined as a subspace of $T_{x_0}M$ by the equation

$$T_{x_0}S = \{\xi \in T_{x_0}M : \langle dF, \xi \rangle = 0\}. \tag{6}$$

Hint. Verify first that every $\xi \in T_{x_0}S$ satisfies $\langle dF, \xi \rangle = 0$.

(b) Let $M = \mathbb{R}^n$. The tangent space $T_{x_0}M$ can be identified with $\mathbb{R}^n$ by using the isomorphism $I : T_{x_0}M \to \mathbb{R}^n$ defined by

$$I(\frac{\partial}{\partial x^i}) = e_i,$$

where $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$. Prove that

$$x_0 + I(T_{x_0}S)$$

is the hyperplane H_{x_0} in $\mathbb{R}^n$ that goes through x_0 and has the normal $\nabla F(x_0)$, where

$$\nabla F = \left(\frac{\partial F}{\partial x^1}, ..., \frac{\partial F}{\partial x^n}\right).$$

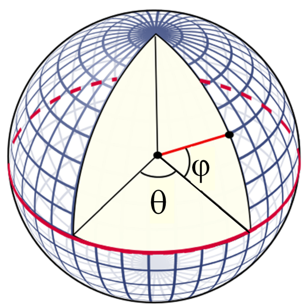
Remark. This result means that the tangent space $T_{x_0}S$ can be naturally identified with the tangent hyperplane H_{x_0} in $\mathbb{R}^n$ to the hypersurface S at the point x_0.

17. For any submanifold S of $\mathbb{R}^n$, denote by $\mathbf{g}_S$ the Riemannian metric on S that is induced by the canonical Euclidean metric

$$\mathbf{g}_{\mathbb{R}^n} = \left(dx^1\right)^2 + ... + \left(dx^n\right)^2.$$

(a) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$. Express the induced metric $\mathbf{g}_{\mathbb{S}^1}$ using the polar angle θ on $\mathbb{S}^1$ as a local coordinate.

(b) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Express the induced metric $\mathbf{g}_{\mathbb{S}^2}$ using the longitude θ and the latitude φ on $\mathbb{S}^2$ as the local coordinates.



18. Let Γ be the graph in $\mathbb{R}^n$ of a C^∞ function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^{n-1}$. Let $\mathbf{g}$ be the canonical metric in $\mathbb{R}^n$, and denote by $\mathbf{g}_\Gamma$ the induced Riemannian metric on Γ considering Γ as a submanifold of $\mathbb{R}^n$. Let $y^1, ..., y^{n-1}$ be the Cartesian coordinates in U that can be regarded as local coordinates on Γ.
Prove that the components of the metric $\mathbf{g}_\Gamma$ in the coordinates $y^1, ..., y^{n-1}$ are as follows:

$$(g_\Gamma)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{7}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Hint. Use the following result from lectures: if S is a submanifold of a Riemannian manifold $(M, \mathbf{g})$ then the induced metric $\mathbf{g}_S$ is given in the local coordinates $x^1, ..., x^n$ on M and $y^1, ..., y^m$ on S by the formula

$$(g_S)_{ij} = g_{kl}\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}. \tag{8}$$

19. Let U be an open set in $\mathbb{R}^m$ and $\Psi : U \to \mathbb{R}^k$ be a smooth mapping. Define the graph Γ of Ψ as follows:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^{m+k} : y = \Psi(x)\right\},$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ and $(x, y) = \left(x^1, ..., x^m, y^1, ..., y^k\right) \in \mathbb{R}^{m+k}$.
Prove that Γ is a submanifold of $\mathbb{R}^{m+k}$ of dimension m.

20. * Let X and Y be smooth manifolds of dimensions n and m, respectively, with $n \geq m$. A mapping $\Phi : Y \to X$ is called smooth if in local coordinates $x^1, ..., x^n$ in X and $y^1, ..., y^m$ in Y it is given by equations

$$x^i = \Phi^i(y^1, ..., y^m), \ i = 1, ..., n,$$

where Φ^i are smooth functions. Let Φ be a smooth mapping as above satisfying the following three properties:

(1) the mapping $\Phi : Y \to X$ is injective;

(2) the rank of the Jacobi matrix $J = \left(\frac{\partial \Phi^i}{\partial y^j}\right)$ of Φ is maximal at all points, that is, it is equal to m;

(3) Φ is a homeomorphism of Y onto its image $S := \Phi(Y) \subset X$.

(a) Prove that S is a submanifold of X of dimension m.

(b) Give examples to show that any of the conditions $(1), (2), (3)$ is essential for this statement.