Матем. Сборник
Том 182 (1991), № 1

Math. USSR Sbornik
Vol. 72 (1992), No. 1

THE HEAT EQUATION ON NONCOMPACT RIEMANNIAN MANIFOLDS

UDC 517.9

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ABSTRACT. The behavior of the Green function $G(x, y, t)$ of the Cauchy problem for the heat equation on a connected, noncompact, complete Riemannian manifold is investigated. For manifolds with boundary it is assumed that the Green function satisfies a Neumann condition on the boundary.

Let M be a geodesically complete, noncompact, smooth, connected Riemannian manifold of dimension n. Let Δ be the Laplace operator (or, equivalently, the Laplace-Beltrami operator) on M. As we know, in local coordinates $x_1, \dots, x_n$ the Laplacian Δ has the form

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right),$$

where the g^{ij} are the contravariant components of the metric tensor (in contrast to the covariant components g_{ij}), and $g = \det \|g_{ij}\|$.

The paper is devoted to the type of estimates of positive solutions of the heat equation

$$u_t - \Delta u = 0 \tag{0.1}$$

(where $u = u(x, t)$, $x \in M$, $t \geq 0$) called Harnack's inequality.

The classical Harnack inequality states that a positive harmonic function $v(x)$ defined in a Euclidean ball B_R of radius R satisfies the estimate

$$\sup_{B_r} v / \inf_{B_r} v \leq P, \tag{0.2}$$

where B_r is a concentric ball of radius $r < R$, while the constant P depends only on the ratio of the radii R/r (and on the dimension of the space).

From (0.2) one can deduce many important properties of harmonic functions, and it is therefore not surprising that much effort has been expended to generalize Harnack's inequality to solutions of elliptic and then parabolic equations. In $\mathbf{R}^n$ it was established by Moser in [4] and [5] for uniformly elliptic and uniformly parabolic equations in divergence form. After that it beame possible to approach directly the question of just what geometric properties of the space entail Harnack's inequality (and also other properties of solutions).

If v is a harmonic function on a Riemannian manifold M (i.e., $\Delta v = 0$) defined and positive in a precompact geodesic ball $B_R \subset M$, then, considering the Laplace equation in local coordinates as a uniformly elliptic equation, one can obtain (0.2). It is true—and this is most essential—that the constant P will depend on R and r, and not just on their ratio.

1991 *Mathematics Subject Classification*. Primary 58G11.

This sort of Harnack inequality, which is naturally called *local* (in contrast to a global inequality when P depends only on R/r), makes it possible to study only local properties of solutions, but not properties such as, for example, Liouville's theorem and others. The same applies to the heat equation (a formulation of the corresponding Harnack inequality is presented in §4).

The purpose of this paper is to obtain the weakest possible conditions on the manifold M under which a (global) Harnack inequality is satisfied for the heat equation (and thus also for the Laplace equation).

Bombieri and Giusti [2] (see also Yau [3]) carried out the first geometric analysis of Moser's proofs. They established that Moser's proofs can be carried over to a manifold M if it satisfies the following conditions:

(a) The ratio of the volumes of any two concentric balls of radii R and $2R$ does not exceed A, where A is the same for all balls.

(b) The first eigenvalue of the Neumann problem in any ball of radius R is not less than a/R^2, where $a > 0$ is the same for all balls (Poincaré's inequality).

(c) For any function $f \in C_0^\infty(M)$

$$\int_M |\operatorname{grad} f| \geq b \left(\int_M |f|^{n/(n-1)} \right)^{(n-1)/n},$$

where $b > 0$ is a constant not depending on f (Sobolev's inequality).

As is known, in $\mathbf{R}^n$ all these conditions are satisfied. Other known proofs of Harnack's inequality in $\mathbf{R}^n$, for example, Landis' proof [13] (see also [24]), actually use the same geometric properties of $\mathbf{R}^n$ but in another form (thus, properties (b) and (c) can be derived from the isoperimetric partition property in a Euclidean ball: if a hypersurface Γ divides a ball into two parts having volume $\geq v$, then $\operatorname{meas}_{n-1}\Gamma \geq c_n v^{(n-1)/n}$, where $c_n > 0$; see [13] and [22]).

One of the basic results of our paper is that Harnack's inequality for equation (0.1) (and thus also for the Laplace equation $\Delta u = 0$) is satisfied if the manifold M satisfies only conditions (a) and (b). Moreover, condition (b) can be relaxed, replacing it by condition (b′) (to be formulated in §1).

Simple examples show that the superfluous condition (c) is not a consequence of (a) and (b). It is not hard to show that in a cylinder $K \times \mathbf{R}^n$, where K is a compact manifold, (a) and (b) are satisfied (this follows from results of [26]), but (c) is not satisfied (since Sobolev's inequality implies growth of the volume of a ball of radius $R \to \infty$ like R^n). Condition (a) alone does not guarantee Harnack's inequality. Corresponding examples have long been known; see, for example, [19]. Violation of Harnack's inequality in these examples occurs due to the presence on the manifold of "narrow" places along which a solution may vary strongly. Condition (b) (and (b′)) forbids just such situations. It remained unclear whether condition (b′) is necessary for Harnack's inequality. As shown in §5, condition (a) follows from Harnack's inequality for the heat equation and is thus a necessary condition.

Condition (b′) is not altogether transparent. In §2 we show that (a) and (b′) result from the following rather graphic geometric condition. We denote by Γ_q^x a homothety of the manifold M along the shortest geodesic with center at the point x and with coefficient $q \in (0, 1)$. Suppose Γ_q^x for $q \in [1/2, 1]$ reduces the volume of any ball by no more than C times, where $C > 1$ does not depend on x, q, or the ball. Then (a) and (b′) are satisfied. For example, this "homothety" condition holds on manifolds of nonnegative Ricci curvature. By the way, on such manifolds Harnack's inequality for the heat equation was proved by another method by Li and Yau [6] (and by Yau for the Laplace equation still earlier [9]).

The structure of the paper is as follows. Some consequences of conditions (a) and (b′) are derived in §1. The "homothety condition" mentioned above is proved in §2. A mean-value theorem used in §4 for the proof of Harnack's inequality is proved in §3. Necessary conditions for Harnack's inequality are proved in §5, and consequences are discussed.

The basic results of the paper were announced in 1987 in [25].

I am grateful to E. M. Landis for fruitful discussions of the questions touched on in the paper, and to A. K. Gushchin for useful remarks.

Notation. $d(x, y)$ is the geodesic distance between points $x, y \in M$; B_R^x is the geodesic open ball with center at the point $x \in M$ and of radius R; $\operatorname{meas}_k A$ is the k-dimensional Riemannian volume of a set A lying in M or in $M \times \mathbf{R}$; and

$$|A| \equiv \operatorname{meas} A \equiv \begin{cases} \operatorname{meas}_n A & \text{if } A \subset M, \\ \operatorname{meas}_{n+1} A & \text{if } A \subset M \times \mathbf{R}; \end{cases}$$

$$f_+ \equiv \tfrac{1}{2}(f + |f|); \qquad f_+^2 \equiv (f_+)^2.$$

All integrals unless otherwise mentioned are taken with respect to the Riemannian measure of M or of the Riemannian product $M \times \mathbf{R}$.

§1. Inequalities of Poincaré type

Everywhere in this paper M denotes a noncompact, smooth, connected Riemannian manifold of dimension n. It may have a boundary ∂M. The manifold M is always assumed to be metrically complete, i.e., any ball B_R^x is a precompact set (in the case of an empty boundary this is equivalent to geodesic completeness).

In this section it is everywhere assumed that the following conditions are satisfied on the manifold M:

For some numbers $A > 0$, $a > 0$, and $N > 1$, and for any $x \in M$ and $R > 0$

(a)

$$|B_{2R}^x| \le A|B_R^x|; \tag{1.1}$$

(b′) For any function $f \in C^\infty(B_{NR}^x)$

$$\int_{B_{NR}^x} |\nabla f|^2 \ge \frac{a}{R^2} \inf_{\xi \in \mathbf{R}} \int_{B_R^x} (f - \xi)^2 \tag{1.2}$$

(as is known, the infimum is achieved when ξ is equal to the arithmetic mean of f in B_R^x).

We observe that if in (1.2) we set $N = 1$, then we obtain precisely Poincaré's inequality (b). As we know, the validity of Poincaré's inequality for domains of Euclidean space depends on the smoothness of the boundary of the domain. Since smoothness of the boundary of a geodesic ball has no direct relation to the geometric properties of the manifold of interest to us, inequality (1.2) under the condition $N > 1$ is a more natural characteristic of the manifold than Poincaré's inequality (b). On the other hand, the fact that the integrals in (1.2) are taken over distinct balls creates considerable technical difficulties for the application of this inequality. Consequences of conditions (1.1) and (1.2) more convenient for applications are presented in the theorems of this section.

Everywhere below const denotes a positive constant depending only on A, a, and N. We suppose that the number N is sufficiently large, for example, $N > 2$.

Theorem 1.1. *For any two intersecting balls B_R^x and B_r^y, where $R \ge r > 0$,*

$$A_2(R/r)^{\alpha_2} \le |B_R^x|/|B_r^y| \le A_1(R/r)^{\alpha_1}, \tag{1.3}$$

where the positive numbers A_1, A_2, α_1, and α_2 depend only on A.

Proof. Let m be an integer such that $2^m \le R/r < 2^{m+1}$. From (1.1) it follows that

$$|B_R^x| \le |B_{2R+r}^y| \le |B_{2^{m+3}r}^y| \le A^{m+3}|B_r^y| \le A^3(R/r)^{\log_2 A}|B_r^y|.$$

Thus, the right inequality in (1.3) is satisfied for $A_1 = A^3$ and $\alpha_1 = \log_2 A$.

Before proving the left inequality, we show that

$$|B_{3R}^x| \ge (1 + A^{-3})|B_R^x|.$$

Indeed, if z is any point a distance $2R$ from x, then, by what has been proved above,

$$\begin{aligned}|B_{3R}^x| &= |B_R^x| + |B_{3R}^x \setminus B_R^x| \ge |B_R^x| + |B_R^z| \\ &\ge |B_R^x| + A^{-3}|B_R^x| = (1 + A^{-3})|B_R^x|.\end{aligned}$$

Suppose, further, that m is an integer such that $3^m \le R/r < 3^{m+1}$. Then

$$\begin{aligned}|B_R^x| &\ge A^{-3}|B_R^y| \ge A^{-3}(1 + A^{-3})^m|B_r^y| \\ &\ge (A^3 + 1)^{-1}(R/r)^{\log_3(1+A^{-3})}|B_r^y|.\end{aligned}$$

Thus, $A_2 = (1 + A^3)^{-1}$ and $\alpha_2 = \log_3(1 + A^{-3})$. Theorem 1.1 is proved.

Theorem 1.2. *For each* $\varepsilon > 0$ *and for each Lipschitz function* f *in the ball* $B_{(1+\varepsilon)R}^z$, *where* $z \in M$ *and* $R > 0$,

$$\int_{B_{(1+\varepsilon)R}^z} |\Delta f_+|^2 \ge \text{const}\frac{\varepsilon^\alpha |H|}{R^2|B_R^z|}\int_{B_R^z} f_+^2, \tag{1.4}$$

where $H = \{f \le 0\} \cap B_R^z$, *and*

$$\int_{B_{(1+\varepsilon)R}^z} |\nabla f|^2 \ge \text{const}\frac{\varepsilon^\alpha}{R^2}\inf_\xi \int_{B_R^z} (f - \xi)^2. \tag{1.5}$$

Here $\alpha > 0$ *depends on* A.

Proof. It obviously suffices to restrict attention to infinitely smooth functions f. We shall first show that (1.4) implies (1.5). Indeed, there exists ξ such that each of the sets $\{f \ge \xi\}$ and $\{f \le \xi\}$ occupies at least half the volume of B_R^z. Applying (1.4) to the functions $f - \xi$ and $\xi - f$ and adding the inequalities thus obtained, we get (1.5).

We proceed to the proof of (1.4).

Lemma 1.1. *Let* $f \in C^\infty(B_{Nr}^x)$, *and suppose that the volumes of the sets* $\{f \le t\} \cap B_r^x$ and $\{f \ge t'\} \cap B_r^x$, *where* $t' > t$, *are equal to* V *and* V' *respectively. Then*

$$\int_{\{t<f<t'\}} |\nabla f|^2 \ge \text{const}\frac{(t'-t)^2 V V'}{r^2|B_r^x|}. \tag{1.6}$$

Proof. We apply inequality (1.2) to the function Φ equal to t in $\{f \le t\}$, to t' in $\{f \ge t'\}$, and to f in $\{t < f < t'\}$. For some ξ we obtain

$$\begin{aligned}\int_{\{t<f<t'\}} |\nabla f|^2 &\ge \int_{B_{Nr}^x} |\nabla\Phi|^2 \ge \frac{a}{R^2}\int_{B_r^z} (\Phi - \xi)^2 \\ &\ge \frac{a}{r^2}(V(t-\xi)^2 + V'(t'-\xi)^2) \ge \frac{a(t'-t)^2 V V'}{r^2(V + V')} \ge \frac{a(t'-t)^2 V V'}{r^2|B_r^x|}.\end{aligned}$$

Here we use the fact that the minimum of the quadratic function of ξ

$$V(t-\xi)^2 + V'(t'-\xi)^2$$

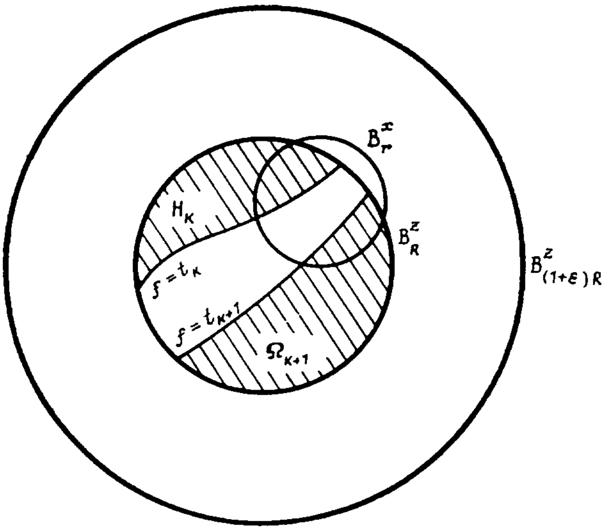

FIGURE 1

is equal to

$$(t' - t)^2 VV'/(V + V').$$

The lemma is proved.

In order to use this lemma we must first decompose the region $\{f > 0\} \cap B^x_{(1+\varepsilon)R}$ into the sets of the form $\{t_k < f < t_{k+1}\}$, and then for each of these sets find a suitable ball in which we can apply (1.6). We set

$$m(t) = \text{meas}\{f > t\} \cap B^z_R, \qquad \overline{m}(t) = \text{meas}\{f \geq t\} \cap B^z_R.$$

We construct an increasing sequence $\{t_k\}$ such that

$$t_0 = 0 \quad \text{and} \quad \overline{m}(t_{k+1}) \geq (1 - \delta)m(t_k), \tag{1.7}$$

where we shall choose $\delta > 0$ later. We fix k and seek a ball B^x_r of radius $r = \varepsilon R/2N$ such that the volumes of the sets $\Omega_k \cap B^x_r$ and $H_k \cap B^x_r$, where

$$\Omega_k = \{f > t_k\} \cap B^z_R, \qquad H_k = \{f \leq t_k\} \cap B^z_R,$$

are sufficiently large (see Figure 1).

We set

$$\mu_1(x) = |B^x_r \cap \Omega_k|/|B^x_r|, \qquad \mu_2(x) = |B^x_r \cap H_k|/|B^x_r|.$$

We introduce the function

$$\mathrm{X}(x, y) = \begin{cases} 1 & \text{if } d(x, y) < r, \\ 0 & \text{if } d(x, y) \geq r, \end{cases}$$

where $x \in B^x_{R+r}$ and $y \in \Omega_k$. Obviously,

$$\int_{B^z_{R+r}} \mathrm{X}(x, y)\, dx = |B^y_r|, \qquad \int_{\Omega_k} \mathrm{X}(x, y)\, dy = \mu_1(x)|B^x_r|.$$

Therefore,

$$\begin{aligned} \int_{\Omega_k} |B^y_r|\, dy &= \int_{\Omega_k} \int_{B^z_{R+r}} \mathrm{X}(x, y)\, dx\, dy \\ &= \int_{B^z_{R+r}} dx \int_{\Omega_k} \mathrm{X}(x, y)\, dy = \int_{B^z_{R+r}} \mu_1(x)|B^x_r|\, dx. \end{aligned}$$

Hence, for some x and y we have

$$\begin{aligned}\mu_1(x) &\geq |B^z_{R+r}|^{-1}|B^x_r|^{-1}\int_{\Omega_k}|B^y_r|\,dy \geq \frac{|B^y_r|\,|Q_k|}{|B^z_{R+r}|\,|B^x_r|}\\ &\geq \frac{|\Omega_k|}{|B^z_R|}\frac{|B^z_R|}{|B^z_{R+r}|}\frac{|B^y_r|}{|B^x_r|} \geq \frac{|\Omega_k|}{|B^z_R|}\frac{|B^z_R|}{|B^z_{2R}|}\frac{|B^y_r|}{|B^x_{2R}|}\\ &\geq A_1^{-2}\left(\frac{r}{4R}\right)^{\alpha_1}\frac{|\Omega_k|}{|B^z_R|} = A_1^{-2}\left(\frac{\varepsilon}{8N}\right)^{\alpha_1}\frac{|\Omega_k|}{|B^z_R|}.\end{aligned} \tag{1.8}$$

In exactly the same way, for some $x \in B^z_{R+r}$ we have

$$\mu_2(x) \geq A_1^{-2}\left(\frac{\varepsilon}{8N}\right)^{\alpha_1}\frac{|H_k|}{|B^z_R|}. \tag{1.9}$$

Since for all $x \in B^z_{R+r}$

$$\mu_1(x) + \mu_2(x) = 1 > A_1^{-2}\left(\frac{\varepsilon}{8N}\right)^{\alpha_1}\frac{|\Omega_k|}{|B^z_R|} + A_1^{-2}\left(\frac{\varepsilon}{8N}\right)^{\alpha_1}\frac{|H_k|}{|B^z_R|},$$

there exists a point x for which (1.8) and (1.9) are satisfied simultaneously. We fix this point and apply Lemma 1.1 in the ball B^x_{Nr} for $t = t_k$ and $t' = t_{k+1}$ (the ball B^x_{Nr} lies in the domain of the function f by the choice of r).

We note that $|H_k| \geq |H|$, and hence

$$\operatorname{meas}\{f \leq t_k\} \cap B^x_r = \mu_2(x)|B^x_r| \geq \text{const}\,\varepsilon^{\alpha_1}\frac{|H|}{|B^z_R|}|B^x_r|. \tag{1.10}$$

Since

$$\begin{aligned}\{f \geq t_{k+1}\} \cap B^x_r \supset \overline{\Omega}_{k+1} \cap B^x_r &= \Omega_k \backslash (\Omega_k \backslash \overline{\Omega}_{k-1}) \cap B^x_r\\ &\supset (\Omega_k \cap B^x_r)\backslash(\Omega_k\backslash\overline{\Omega}_{k+1}),\end{aligned}$$

$$|\Omega_k| = m(t_k), \quad |\overline{\Omega}_{k+1}| = \overline{m}(t_{k+1}), \quad |\Omega_k\backslash\overline{\Omega}_{k+1}| \geq \delta m(t_k),$$

and $\operatorname{meas}(\Omega_k \cap B^k_r) = \mu_1(x)|B^x_r|$, by (1.8)

$$\operatorname{meas}\{f \geq t_{k+1}\} \cap B^x_r \geq \text{const}\,\varepsilon^{\alpha_1}\frac{m(t_k)}{|B^z_R|}|B^x_r| - \delta m(t_k).$$

We note that

$$\frac{|B^x_r|}{|B^z_R|} \geq A_1^{-1}\left(\frac{r}{R}\right)^{\alpha_1} = A_1^{-1}\left(\frac{\varepsilon}{2N}\right)^{\alpha_1}.$$

Setting $\delta = \frac{1}{2}\text{const}\,\varepsilon^{\alpha_1}A_1^{-1}(\varepsilon/2N)^{\alpha_1}$, we obtain $\operatorname{meas}\{f \geq t_{k+1}\} \cap B^x_r \geq \delta m(t_k)$. According to Lemma 1.1, from this and (1.10) we have

$$\int_{\{t_k<f<t_{k+1}\}}|\nabla f|^2 \geq \text{const}\frac{(t_{k+1}-t_k)^2}{R^2}\varepsilon^{\alpha_1}\frac{|H|}{|B^z_R|}\delta m(t_k).$$

Summing over all k and replacing δ by its value, we obtain

$$\int_{B^z_{(1+\varepsilon)R}}|\nabla f_+|^2 \geq \text{const}\frac{\varepsilon^{3\alpha_1}}{R^2}\frac{|H|}{|B^z_R|}\sum_{k=0}^{\infty}(t_{k+1}-t_k)^2 m(t_k). \tag{1.11}$$

On the other hand, if $m(t_k) \to 0$, then there is the obvious inequality

$$\int_{B^z_R} f_+^2 \leq \sum_{k=0}^{\infty} t_{k+1}^2(m(t_k) - m(t_{k+1})). \tag{1.12}$$

We now compare the sums on the right sides of (1.11) and (1.12). For this we specify the choice of the sequence $\{t_k\}$. We recall that so far we have required of it only that (1.7) be satisfied. We set

$$t_0 = 0, \qquad t_{k+1} = \inf\{t: \overline{m}(t) \geq (1-\delta)m(t)\}, \quad k = 0, 1, 2, \dots.$$

Obviously condition (1.7) is satisfied, but together with it we also have

$$m(t_{k+1}) \leq (1-\delta)m(t_k). \tag{1.13}$$

In particular, this implies that $m(t_k) \to 0$ as $k \to \infty$. We now use a lemma.

Lemma 1.2. *Let $\{t_k\}$ be an increasing sequence with $t_0 = 0$, and let $\{m_k\}$ be a decreasing sequence of positive numbers for which $m_{k+1} \leq (1-\delta)m_k$, where $\delta > 0$. Then*

$$\sum_{k=0}^{\infty}(t_{k+1}-t_k)^2 m_k \geq \frac{\delta}{12}\sum_{k=0}^{\infty} t_{k+1}^2(m_k - m_{k+1}).$$

Proof. On the positive semiaxis we consider a piecewise linear function $\varphi(\mu)$ defined as follows: $\varphi(m_k) = t_k$ $\varphi(\mu)$ is linear on each interval (m_{k+1}, m_k), and $\varphi(\mu) = 0$ on $(m_0, +\infty)$. We remark that on (m_{k+1}, m_k)

$$\frac{d\varphi}{d\mu} = \frac{t_{k+1}-t_k}{m_k - m_{k+1}}.$$

Using (1.13), we therefore have

$$\begin{aligned}\sum_{k=0}^{\infty}(t_{k+1}-t_k)^2 m_k &= \sum_{k=0}^{\infty}\frac{(t_{k+1}-t_k)^2}{(m_k-m_{k+1})^2}(m_k-m_{k+1})^2 m_k \\ &\geq \sum_{k=0}^{\infty}\frac{(t_{k+1}-t_k)^2}{(m_k-m_{k+1})^2}(m_k-m_{k+1})\delta m_k^2 \geq \delta\int_0^{\infty}\varphi'^2\mu^2\,d\mu.\end{aligned}$$

On the other hand, since the integral over the segment $[m_{k+1}, m_k]$ of the quadratic function $\varphi(\mu)^2$ monotone on this segment is not less than $1/3$ the product of the length of the segment by the maximum t_{k+1}^2 of the function in question (on this segment), it follows that

$$\sum_{k=0}^{\infty} t_{k+1}^2(m_k-m_{k+1}) \leq 3\sum_{k=0}^{\infty}\int_{m_{k+1}}^{m_k}\varphi(\mu)^2\,d\mu = 3\int_0^{\infty}\varphi(\mu)^2\,d\mu.$$

Finally, according to Hardy's inequality

$$\int_0^{\infty}\varphi'^2\mu^2\,d\mu \geq \frac{1}{4}\int_0^{\infty}\varphi(\mu)^2\,d\mu.$$

Collecting all these inequalities, we complete the proof of the lemma.

From (1.11), (1.12), and Lemma 1.2 we obtain (1.4) in an obvious manner. Theorem 1.2 is proved.

Theorem 1.3. *Let f be a Lipschitz function in the ball B_R^x, and let $H = \{y \in B_{R/2}^x: f(y) \leq 0\}$. Then*

$$\int_{B_R^x}|\nabla f_+|^2\eta^2 \geq \text{const}\frac{|H|}{R^2|B_R^x|^2}\left\{\int_{B_R^x} f_+\eta^2\right\}^2, \tag{1.14}$$

where $\eta(y) = (d(y)/R)^{\alpha/2}$, α *is the constant of Theorem* 1.2, *and* $d(y)$ *is the distance from the point* y *to* ∂B_R^x.

Proof. We set $R_k = R(1-2^{-k})$, $k \geq 1$. According to Theorem 1.2, for each pair of balls $B_{R_k}^x$, $B_{R_{k+1}}^x$ we have

$$\begin{aligned}\int_{B_{R_{k+1}}^x} |\nabla f_+|^2 &\geq \text{const}\left[\frac{R_k - R_{k+1}}{R}\right]^\alpha \frac{|H|}{R^2|B_R^x|}\int_{B_{R_k}^x} f_+^2 \\ &\geq \text{const}\frac{2^{-\alpha k}|H|}{R^2|B_R^x|}\int_{B_{R_k}^x} f_+^2.\end{aligned} \tag{1.15}$$

We set $\Omega_k = B_{R_k}^x \backslash B_{R_{k-1}}^x$, $k > 1$, and $\Omega_1 = B_{R_1}^x$. Multiplying (1.15) by $2^{-\alpha k}$ and summing over all $k = 1, 2, \ldots,$ we obtain

$$\sum_{k=1}^\infty 2^{-\alpha k}\int_{\Omega_k} |\nabla f_+|^2 \geq \text{const}\frac{|H|}{R^2|B_R^x|}\sum_{k=1}^\infty 4^{-\alpha k}\int_{\Omega_k} f_+^2. \tag{1.16}$$

Since for $2R\cdot 2^{-k} \geq d(y) \geq R\cdot 2^{-k}$ for $y \in \Omega_k$, it follows that $2^{-k\alpha} \leq (d(y)/R)^\alpha = \eta^2$ and $4^{-k\alpha} \leq (d(y)/2R)^{2\alpha} = \eta^4/2^{2\alpha}$. Therefore, from (1.16) it follows that

$$\int_{B_R^x} |\nabla f_+|^2\eta^2 \geq \text{const}\frac{|H|}{R^2|B_R^x|}\int_{B_R^x} f_+^2\eta^4.$$

Applying the Cauchy-Schwarz-Bunyakovskiĭ inequality to the right side of this relation, we obtain (1.14).

Theorem 1.4. *For any* $y \in M$ *and* $R > 0$, *for any domain* Ω, $\overline{\Omega} \subset B_R^y$, *and for any Lipschitz function* u *in* $\overline{\Omega}$ *vanishing on* $\partial\Omega$,

$$\int_\Omega |\nabla u|^2 \geq \frac{b}{R^2}\left\{\frac{|B_R^x|}{|\Omega|}\right\}^\beta \int_\Omega u^2, \tag{1.17}$$

where $b > 0$ *and* $\beta > 0$ *depend only on* A, a, *and* N.

Before proving the theorem we observe that it admits the following reformulation. If $\lambda_1(\Omega)$ denotes the first eigenvalue of the boundary value problem

$$\Delta u + \lambda u = 0, \quad u|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial M\cap\Omega} = 0, \tag{1.18}$$

where ν is the normal to the boundary ∂M (in the case of an empty boundary ∂M this is the Dirichlet problem; in the general case we also call (1.18) the Dirichlet problem), it follows that

$$\lambda_1(\Omega) \geq \frac{b}{R^2}\left\{\frac{|B_R^x|}{|\Omega|}\right\}^\beta. \tag{1.19}$$

For example, in $\mathbf{R}^n$ (1.19) is satisfied for $\beta = 2/n$.

Proceeding to the proof of (1.17), for each $t \geq 0$ we consider the set $V_t = \{u > t\}$, and define $m(t) = \operatorname{meas} V_t$ and $\overline{m}(t) = \operatorname{meas}\overline{V}_t$. We fix some $t > 0$ and suppose that $t' > t$ is such that

$$\overline{m}(t') \geq (1-\delta)m(t), \tag{1.20}$$

where $\delta > 0$ will be chosen later. For each point $x \in V_t$ we construct a ball B_r^x such that

$$\operatorname{meas}(B_r^x \cap V_t) = \tfrac{1}{2}|B_r^x|. \tag{1.21}$$

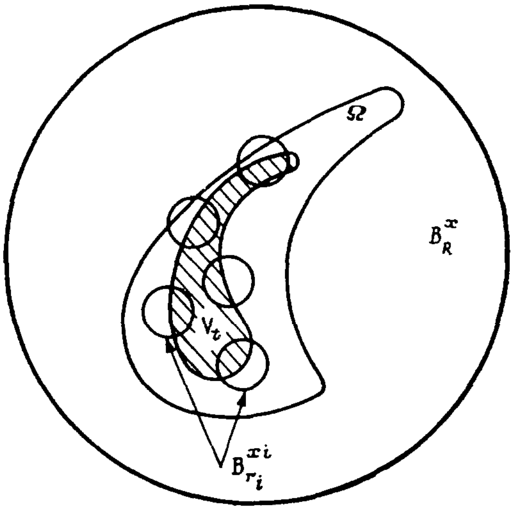

FIGURE 2

Such an r (depending, of course, on x) exists, since as $r \to 0$ the left side of (1.21) is greater than the right side, while for $r > R$, according to Theorem 1.1,

$$\operatorname{meas}(B_r^x \cap V_t) \leq |B_R^y| \leq A_2^{-1}(R/r)^{\alpha_2}|B_r^x|.$$

Therefore, if r is so large that

$$A_2^{-1}(R/r)^{\alpha_2} < \tfrac{1}{2},$$

then the right side in (1.21) is greater than the left side. Hence (1.21) is satisfied for some r such that

$$r \leq \operatorname{const} R. \tag{1.22}$$

The collection of all balls B_r^x covers V_t. From them we select a countable (or finite) number $B_{r_i}^{x_i}$ such that the balls $B_{2r_i}^{x_i}$ do not intersect, while the balls $B_{10r_i}^{x_i}$ cover V_t (see [12], p. 272). By (1.1)

$$\sum_i |B_{r_i}^{x_i}| \geq A^{-4} \sum_i |B_{10r_i}^{x_i}| \geq A^{-4} m(t),$$

and by (1.21) the volume of that part of V_t covered by the balls $B_{r_i}^{x_i}$ is not less than $\frac{1}{2}A^{-4}m(t)$ (see Figure 2).

Suppose the function v is equal to $t' - t$ and $V_{t'}$, to $u - t$ in $V_t \backslash V_{t'}$, and to 0 outside V_t. Applying inequality (1.4) for the function v in the balls $B_{r_i}^{x_i}$ and $B_{2r_i}^{x_i}$, and considering that the set $\{v \leq 0\}$ occupies half of the volume in the ball $B_{r_i}^{x_i}$, we obtain

$$\int_{B_{2r_i}}^{x_i} \geq \frac{\text{const}}{r_i^2} \int_{B_{r_i}^{x_i}} v^2.$$

Adding these inequalities over all i and noting that

$$|B_{r_i}^{x_i}| \geq |B_{r_i/\text{const}}^{x_i}| \geq \operatorname{const}(r_1/R)^{\alpha_1}|B_R^y|$$

(where const on the left side is the same as in (1.22)), i.e.,

$$r_i \leq \operatorname{const} R(|B_{r_i}^{x_i}|/|B_R^y|)^{1/\alpha_1} \leq \operatorname{const} R(|\Omega|/|B_R^y|)^{1/\alpha_1},$$

we obtain

$$\int_{V_t\setminus V_{t'}} |\nabla u|^2 = \int_M |\nabla v|^2 \geq \frac{\text{const}}{R^2}\left\{\frac{|B_R^y|}{|\Omega|}\right\}^{2/\alpha_1} \int_{\bigcup_i B_{r_i}^{x_i}} v^2.$$

Since the balls $B_{r_i}^{x_i}$ cover in V_t a volume at least $\frac{1}{2}A^{-4}m(t)$, in $\overline{V}_{t'}$ they cover (by (1.20)) a volume at least $\frac{1}{2}A^{-4}m(t) - \delta m(t)$, which is not less than $\delta m(t)$ for $\delta = \frac{1}{4}A^{-4}$. Since $v|_{\overline{V}_{t'}} = t' - t$, it follows that

$$\int_{\bigcup_i B_{r_i}^{x_i}} v^2 \geq (t'-t)^2 \delta m(t),$$

and so

$$\int_{V_t\setminus V_{t'}} |\nabla u|^2 \geq \frac{\text{const}}{R^2}\left\{\frac{|B_R^y|}{|\Omega|}\right\}^{2/\alpha_1} (t'-t)^2 m(t). \tag{1.23}$$

On the other hand, obviously,

$$\int_{V_t\setminus V_{t'}} u^2 \leq t'^2(m(t) - m(t')). \tag{1.24}$$

Finally, in analogy to what was done in the proof of Theorem 1.2, we choose a sequence $\{t_k\}$ satisfying condition (1.7). Setting $t = t_k$ and $t' = t_{k+1}$ in (1.23) and (1.24), summing over k, and using Lemma 1.2, we obtain (1.17).

§2. Manifolds with a "homothety condition"

In this section we assume that on the manifold M a homothety is defined in the following manner. Suppose any two points $x, y \in M$ are joined by a piecewise smooth, non-self-intersecting curve $\gamma_{x,y}$ (where $\gamma_{x,y} = \gamma_{y,x}$). A *homothety with center at the point* $z \in M$ *and coefficient* q, $0 < q < 1$, is a mapping $\Gamma_q^z: M \to M$ given by

$$\Gamma_q^z(x) = \gamma_{x,y}(q\sigma),$$

where $\gamma_{z,x}(t): [0, \sigma] \to M$ is the natural parametrization of the curve γ with $\gamma_{z,x}(0) = z$ and $\gamma_{z,x}(\sigma) = x$. We also assume that this homothety satisfies the following conditions:

1) If $z \in \gamma_{x,y}$, then $\gamma_{x,z} \subset \gamma_{x,y}$.

2) For some constant $N \geq 1$ and all $x, y \in M$

$$\operatorname{meas}_1 \gamma_{x,y} \leq N d(x, y). \tag{2.1}$$

3) For any point $z \in M$, any $q \in (1/2, 1)$, and any bounded domain $B \subset M$ the image $\Gamma_q^z(B)$ is a measurable set and

$$\operatorname{meas} \Gamma_q^z(B) \geq c|B|, \tag{2.2}$$

where $c > 0$ is a constant.

Examples. 1. Let M be a convex unbounded domain in Euclidean space. If $\gamma_{x,y}$ is a line segment joining the points x and y, then conditions 1)–3) are obviously satisfied with constants $c = 2^{-n}$ and $N = 1$.

2. Let $\gamma_{x,y}$ be a segment of the shortest geodesic on an arbitrary manifold without boundary (if the points x and y can be joined by several shortest ones, we choose one of them arbitrarily). Conditions 1) and 2) are satisfied by the general properties of geodesics. Condition 3) is a strong condition on the geometry of the manifold. It is easy to see, for example, that in Lobachevsky space and, more generally, on

Cartan-Hadamard manifolds with negative curvature bounded away from zero it is not satisfied (at any rate because on such manifolds the volume of a ball of radius R grows exponentially as $R \to \infty$, while condition (2.2) implies power growth). As Sullivan [20] and Anderson [21] have proved, on such manifolds there exists a nontrivial bounded harmonic function, and thus Harnack's inequality for the heat and Laplace equations is not satisfied.

3. To the contrary, we shall prove that if the manifold M has nonnegative Ricci curvature, then condition (2.2) is satisfied. Suppose first that the domain B lies away from a cut site S of the point z. We set $B_\tau = \Gamma_\tau^z(B)$. For each point $x \notin S$ the curve $\gamma(\tau) \equiv \Gamma_\tau^z(x)$ is a phase curve of the variable vector field $(1/\tau)d(x)\nabla d(x)$, where $d(x) \equiv d(x, z)$. By the Liouville-Ostrogradskiĭ formula we have

$$\frac{d}{d\tau}|B_\tau| = \int_{B_\tau} \operatorname{div}\left(\frac{d\nabla d}{\tau}\right) \leq \int_{B_\tau} \frac{1 + d\Delta d}{\tau}.$$

If the Ricci curvature is nonnegative, then $\Delta d \leq (n-1)/d$ (see [23]), so that $(d/d\tau)|B_\tau| \leq (n/\tau)|B_\tau|$, $|B| \leq \tau^{-n}|B_\tau|$, and it is possible to set $c = 2^{-n}$.

If B intersects a cut site, then we have

$$|B| = |B\backslash S| \leq \tau^{-n}\operatorname{meas}\Gamma_\tau^z(B\backslash S) = \tau^{-n}\operatorname{meas}\Gamma_\tau^z(B),$$

since $|S| = 0 = \operatorname{meas}\Gamma_\tau^z(S)$. Indeed, S is the image of the exponential mapping of a cut site $\widetilde{S}$ in the tangent space T_zM, while $\widetilde{S}$ has measure zero as the graph of a function on the unit sphere (see [16], p. 100). The same applies to $\Gamma_\tau^z(S)$.

The following theorem is the main result of this section.

Theorem 2.1. *If on the manifold M there is a homothety satisfying conditions* 1)–3), *then conditions* (a) *and* (b′) *of* §1 *are also satisfied.*

Proof. We first note that inequality (2.2) extends in an obvious way to all $q \in (0, 1)$:

$$C(q)\operatorname{meas}\Gamma_q^z(B) \geq |B|, \tag{2.3}$$

where $C(q) = c^{-k}$, and k is the smallest positive integer such that $q > (1/2)^k$ (the composition of k homotheties with coefficient $q^{1/k} > 1/2$ is equal to one homothety with coefficient q).

We first prove (a). According to (2.3), for any ball B_R^x we have

$$C(q)\operatorname{meas}\Gamma_q^x(B_{2R}^x) \geq |B_{2R}^x|. \tag{2.4}$$

We set $q = 1/2N$. From (2.1) it follows that $\Gamma_q^x(B_{2R}^x) \subset B_R^x$, while condition (a) with coefficient $A = C(q)$ follows from (2.4).

We proceed to the derivation of (b′). We denote by const a positive constant depending only on N and c. We first prove that, for any ball B_{2NR}^z and any function f smooth in it,

$$\int_{B_{3NR}^z} |\nabla f|^2 \geq \frac{\text{const}}{R^2|B_R^z|}\int_{B_R^z}\int_{B_R^z} |f(x) - f(y)|^2\,dx\,dy. \tag{2.5}$$

We note that if $x, y \in B_R^z$, then by (2.1) $\operatorname{meas}_1 \gamma_{x,y} \leq 2NR$, so that in any case $\gamma_{x,y} \subset B_{3NR}^z$. Since

$$|f(x) - f(y)|^2 \leq \left\{\int_{\gamma_{x,y}} |\nabla f|\right\}^2 \leq 2NR\int_{\gamma_{x,y}} |\nabla f|^2,$$

setting $F = |\nabla f|^2$, we find that in place of (2.5) it suffices to prove

$$\int_{B_{3NR}^z} F \geq \frac{\text{const}}{R|B_R^z|}\int_{B_R^z}\int_{B_R^z}\left\{\int_{\gamma_{x,y}} F\right\}dx\,dy. \tag{2.6}$$

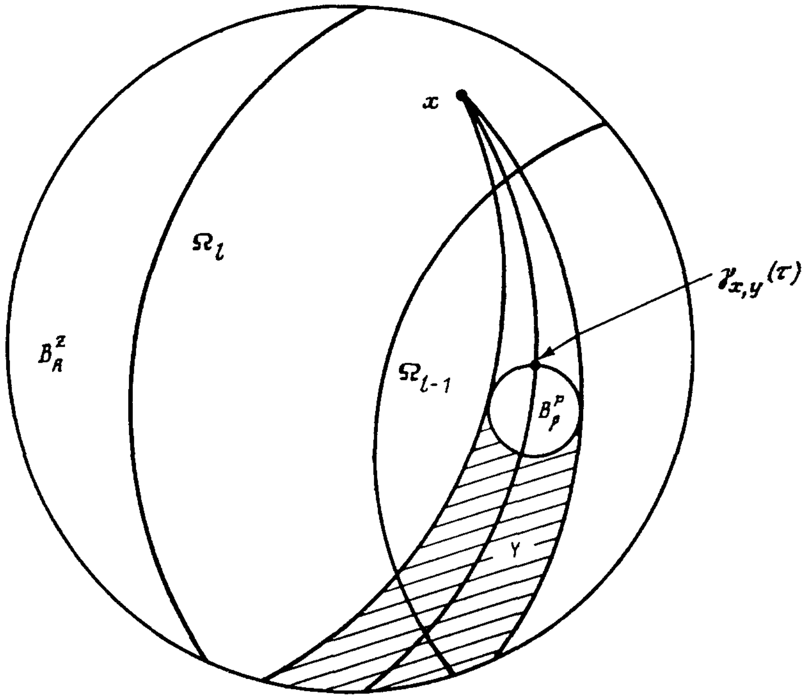

FIGURE 3

By the linearity and continuity in F of both sides of (2.6), it suffices to prove (2.6) for the case where F is the characteristic function of a sufficiently small ball $B_\rho^p \subset B_{3NR}^z$. We denote by $\mathrm{X}(x, y)$ the function equal to 1 if $\gamma_{x,y}$ intersects the ball B_ρ^p and equal to zero otherwise. It is obvious that

$$\int_{\gamma_{x,y}} F \leq 2N\rho\mathrm{X}(x, y).$$

Therefore, the integral on the right side of (2.6) does not exceed

$$2N\rho \int_{B_R^z} \int_{B_R^z} \mathrm{X}(x, y)\, dx\, dy. \tag{2.7}$$

We decompose the region of integration in (2.7) into two parts. We set

$$\Omega_l = B_{\rho 2^l}^p \cap B_R^z, \qquad l = 1, 2, \ldots.$$

Let m be the least integer such that $\Omega_m = B_R^z$. Then, in particular, $2^m\rho \leq 12NR$. Obviously the direct product $B_R^z \times B_R^z$ is the union of sets of the form

$$(\Omega_l \backslash \Omega_{l-1}) \times \Omega_l, \quad \Omega_l \times (\Omega_l \backslash \Omega_{l-1}), \quad 2 \leq l \leq m, \quad \text{and} \quad \Omega_1 \times \Omega_1.$$

We shall estimate the integral of $\mathrm{X}(x, y)$ over $(\Omega_l \backslash \Omega_{l-1}) \times \Omega_l$, $2 \leq l \leq m$. Let $x \in \Omega_l \backslash \Omega_{l-1}$, and let $Y \equiv Y(x) \equiv \{y \in \Omega_l : \gamma_{x,y} \text{ intersects } B_\rho^p\}$. We shall obtain an upper bound for $|Y|$. Let t be the natural parameter on the curve $\gamma_{x,y}$, where $y \in Y$, with $\gamma_{x,y}(0) = x$. For brevity we set $2^l\rho = r$. Since $x, y \in B_r^p$, it follows that $d(x, y) \leq 2r$ and $\text{meas}_1 \gamma_{x,y} \leq 2Nr$. Let τ be the first point of entry of the curve $\gamma_{x,y}$ into the ball $\overline{B}_\rho^p$ (see Figure 3). If the homothety Γ_q^x with coefficient $q = 2Nr/(2Nr + \rho) \geq \tau/(\tau + \rho)$ is applied several times, then any point $\gamma_{x,y}(t)$ for

$t > \tau$ finally lands in the segment $[\tau, \tau+\rho]$ of the curve $\gamma_{x,y}$. The largest number of homotheties needed for this does not exceed

$$\operatorname{meas}_1 \gamma_{x,y}/(\tau q^{-1} - \tau) \leq 16N^2 r/\rho.$$

We have here used the fact that

$$\tau \geq d(x, B^p_\rho) \geq r/2 - \rho = r/2 - 2^{-l} r \geq r/4,$$

since $x \in \Omega_l \backslash \Omega_{l-1}$.

For each $k \geq 0$ we denote by Y_k the set of points $y \in Y$ which after k applications of the homothety Γ^x_q land in the segment $[\tau, \tau+\rho]$ of the curve $\gamma_{x,y}$. Obviously,

$$|Y| \leq \sum_{k \leq 16N^2 r/\rho} |Y_k|.$$

According to (2.3),

$$|Y_k| \leq C(q^k) \operatorname{meas} \Gamma^x_{q^k}(Y_k) \leq C(q^k)|Y_0|.$$

Since

$$q^k \geq \left(\frac{2Nr}{2Mr+\rho}\right)^{16N^2 r/\rho} = \left(1 + \frac{\rho}{2Nr}\right)^{-16N^2 r/\rho} \geq e^{-8N},$$

it follows that $C(q^k) \leq C(e^{-8N})$ and $|Y_k| \leq C(e^{-8N})|Y_0|$. Further, if $y \in Y_0$, then $d(x, y) \leq 2\rho$, so that $|Y_0| \leq |B^p_{2\rho}| \leq A|B^p_\rho|$. Combining these inequalities, we obtain

$$|Y| \leq 17N^2 \frac{r}{\rho} C(e^{-8N}) A |B^p_\rho|.$$

Finally, noting that $|\Omega_l \backslash \Omega_{l-1}| \leq |B^z_R|$, we get

$$\int_{\Omega_l \backslash \Omega_{l-1}} \int_{\Omega_l} \mathrm{X}(x, y)\, dx\, dy \leq \int_{\Omega_l \backslash \Omega_{l-1}} |Y(x)|\, dx \leq 17N^2 2^l C(e^{-8N}) A |B^p_\rho|\, |B^z_R|.$$

The analogous integral over $\Omega_l \times (\Omega_l \backslash \Omega_{l-1})$ can be estimated in the same way. For the integral over $\Omega_1 \times \Omega_1$ we obviously have

$$\int_{\Omega_1} \int_{\Omega_1} \mathrm{X}(x, y)\, dx\, dy \leq |B^p_{2\rho}|\, |B^z_R| \leq A |B^p_\rho|\, |B^z_R|.$$

Adding all the estimates of the integrals of $\mathrm{X}(x, y)$ and noting that $2^1 + 2^2 + \cdots + 2^m < 2^{m+1} \leq 24NR/\rho$, we obtain

$$\int_{B^z_R} \int_{B^z_R} \mathrm{X}(x, y)\, dx\, dy \leq 1000 N^3 \frac{R}{r} C(e^{-8N}) A |B^p_\rho|\, |B^z_R|.$$

Noting that the integral on the right side of (2.6) does not exceed the expression (2.7), which we actually just estimated, while the integral on the left side is equal to $|B^p_\rho|$, we obtain the desired estimate (2.6).

We shall now prove that

$$\int_{B^z_{3NR}} |\nabla f|^2 \geq \frac{\text{const}}{R^2} \inf_\xi \int_{B^z_R} (f - \xi)^2\, dx, \tag{2.8}$$

i.e., condition (b′). For this we find ξ such that each of the sets $\Omega_+ = \{f \geq \xi\} \cap B^z_R$ and $\Omega_- = \{f \leq \xi\} \cap B^z_R$ has volume $\geq \frac{1}{2}|B^z_R|$. Since

$$\begin{aligned}\int_{B^z_R} \int_{B^z_R} |f(x) - f(y)|^2\, dx\, dy &\geq \int_{\Omega_-} \int_{\Omega_+} |f(x) - f(y)|^2\, dx\, dy \\ &\geq |\Omega_-| \int_{\Omega_+} (f(x) - \xi)^2\, dx\end{aligned}$$

and in the same way

$$\int_{B_R^z}\int_{B_R^z} |f(x)-f(y)|^2\,dx\,dy \ge |\Omega_+| \int_{\Omega_-} (f(x)-\xi)^2\,dx,$$

adding these inequalities, we obtain

$$\int_{B_R^z}\int_{B_R^z} |f(x)-f(y)|^2\,dx\,dy \ge \frac{1}{4}|B_R^z| \int_{B_R^z} (f(x)-\xi)^2\,dx.$$

From this and (2.5) we obtain (2.8). The theorem is proved.

§3. A MEAN-VALUE THEOREM

We say that an isoperimetric inequality with function $\Lambda(v)$ (where Λ is a positive, continuous, monotone decreasing function on $(0, +\infty)$) is satisfied in a region $\Omega \subset M$ if for any open set D, $\overline{D} \subset \Omega$, we have

$$\lambda_1(D) \ge \Lambda(|D|), \tag{3.1}$$

where $\lambda_1(D)$ is the first eigenvalue of the Dirichlet problem in D (see (1.18)). For example, in $\mathbf{R}^n$ and on Cartan-Hadamard manifolds there is an isoperimetric inequality with the function

$$\Lambda(v) = av^{-2/n}, \tag{3.2}$$

where $a = a(n) > 0$ (this follows from results of [7] and [18]).

On any manifold M, $\inf \lambda_1(D)$ over all domains $D \subset M$ is equal to the spectral radius of M, and we denote it by $\lambda_1(M)$. As is known (see [8]), on a simply connected manifold with sectional curvature $\le -k^2 < 0$ we have $\lambda_1(M) \ge (n-1)^2k^2/4$, so that for such manifolds we can set

$$\Lambda(v) = \max(av^{-2/n}, (n-1)^2k^2/4). \tag{3.3}$$

If M is a manifold of nonnegative Ricci curvature, then, as follows from Theorems 2.1 and 1.4, in each ball $|B_R^x|$ there is an isoperimetric inequality with function

$$\Lambda(v) = \frac{b}{R^2}\left\{\frac{|B_R^x|}{v}\right\}^\beta, \tag{3.4}$$

where $b, \beta > 0$ depend only on n.

Before formulating the main results of this section, we introduce some notation. The function $v/\Lambda(v)$ is obviously strictly monotonically increasing on $(0, +\infty)$ with range $(0, +\infty)$. It therefore has an inverse function on $(0, +\infty)$, which we denote by ω. We define functions $V(t)$ and $W(r)$ (where $t > 0$ and $r > 0$) by the equalities

$$ct = \int_0^{V(t)} \frac{d\xi}{\omega(\xi)}, \qquad cr = \int_0^{W(r)} \frac{d\xi}{\sqrt{\xi\omega(\xi)}}, \tag{3.5}$$

where $c > 0$ is an absolute constant which will be determined in the course of the proof. Everywhere below we assume that the integrals in (3.5) converge to zero. This is clearly the case if in a neighborhood of zero we have $\Lambda(v) \ge v^{-\varepsilon}$, $\varepsilon > 0$.

Theorem 3.1. *Suppose in some ball B_R^z there is an isoperimetric inequality with function $\Lambda(v)$. Let $Ц = B_R^z \times (0, T)$, $T > 0$, and suppose that in the cylinder $Ц$ a function $u \in C^\infty(\overline{Ц})$ satisfies the inequality*

$$u_t - \Delta u \le 0 \tag{3.6}$$

and the Neumann condition on the boundary of the manifold (if it is nonempty)

$$\partial u/\partial\nu|_{x\in\partial M\cap B_R^z} = 0. \tag{3.7}$$

Then

$$u(z, T)_+^2 \le \frac{4}{\min(V(T), W(R))}\int_{Ц} u_+^2. \tag{3.8}$$

Examples. 1. If $\Lambda(v)$ is the function (3.2), then

$$V(t) = C_1(n)a^{n/2}t^{(n+2)/2}, \qquad W(r) = C_2(n)a^{n/2}r^{n+2},$$

and (3.8) acquires the form

$$u(z, T)_+^2 \le \frac{C(n)a^{-n/2}}{\min(\sqrt{T}, R)^{n+2}}\int_{Ц} u_+^2. \tag{3.9}$$

In $\mathbf{R}^n$ for $T = R^2$ this inequality was proved by Moser [5].

2. If $\Lambda(v) = \max(av^{-2/n}, A)$, $A > 0$, then

$$V(t) \asymp a^{n/2}\min(t, A^{-1})^{(n+2)/2}\exp(cAt),$$
$$W(r) \asymp a^{n/2}\min(r^2, A^{-1})^{(n+2)/2}\exp(c\sqrt{A}r),$$

where the symbol $\asymp$ means "is in finite ratio with", and the constants bounding the ratio of right and left sides in these relations depend only on n.

3. If $\Lambda(v)$ is the function (3.4), then we essentially have the situation of Example 1 for $a = b|B_R^z|^\beta/R^2$ and $n = 2/\beta$. Substituting into (3.9) and noting that $|B_R^z|T = |Ц|$, we obtain

$$u(z, T)_+^2 \le \frac{C(\beta)b^{-1/\beta}}{\min((T/R^2)^{1/\beta}, R^2/T)}\frac{1}{|Ц|}\int_{Ц} u_+^2. \tag{3.10}$$

For $T = R^2$ the coefficient in front of the integral depends on T and R. Inequality (3.10) is the reason why we called Theorem 3.1 a mean-value theorem.

Lemma 3.1. *Suppose, under the conditions of Theorem* 3.1, *that* $v = (u-\theta)_+$, *where* $\theta > 0$ *is an arbitrary number. Let* $\eta(x, t)$ *be a Lipschitz function in* $M \times [0, +\infty)$ *equal to zero for* $t = 0$ *and having support in* $\overline{B_R^z}$ *for each* $t > 0$. *Then*

$$\int_{B_R^z} v^2\eta^2(x, T)dx + \frac{1}{2}\int_{Ц}|\nabla(v\eta)|^2 \le 5\int_{Ц} v^2(|\nabla\eta|^2 + |\eta\eta_t|). \tag{3.11}$$

Proof. We first prove that for any function $\varphi \in C_0^\infty(B_R^z)$, for each $t \in (0, T)$,

$$\int_{B_R^z} vv_t\varphi^2 \le -\int_{B_R^z}(\nabla(v\varphi^2), \nabla v). \tag{3.12}$$

Indeed, if θ is a regular value of the functions u and $u|_{\partial M}$, then from (3.6) and (3.7) we obtain

$$\begin{aligned}\int_{B_R^z} vv_t\varphi^2 &= \int_{\{u>\theta\}} vu_t\varphi^2 \le \int_{\{u>\theta\}}\varphi^2 v\Delta u \\ &= \int_{\{u=\theta\}}\frac{\partial u}{\partial\nu}v\varphi^2 + \int_{\partial M\cap B_R^z}\frac{\partial u}{\partial\nu}v\varphi^2 - \int_{\{u>\theta\}}(\nabla u, \nabla(v\varphi^2)) \\ &= -\int_{B_R^z}(\nabla u, \nabla(v\varphi^2))\end{aligned}$$

(here, in particular, we have used the fact that $v = 0$ on the hypersurface $\{u = \theta\}$). This same argument goes through if θ does not lie in the range of u. If θ is a nonregular value, then (3.12) can be established by passing to the limit from regular values. We note that by this limiting procedure (3.12) extends also to Lipschitz functions φ with support in $\overline{B_R^z}$.

Setting $\varphi(x) = \eta(x, t)$ in (3.12) and integrating on t, we obtain

$$\int_{Ц} v v_t \eta^2 \le -\int_{Ц} |\nabla v|^2 \eta^2 - 2\int_{Ц} (\nabla v, \nabla \eta) v\eta.$$

Noting that

$$-2(\nabla v, \nabla\eta)v\eta \le \tfrac{1}{2}|\nabla v|^2\eta^2 + 2v^2|\nabla\eta|^2,$$
$$|\nabla v|^2\eta^2 \ge \tfrac{1}{2}|\nabla(v\eta)|^2 - |\nabla\eta|^2 v^2.$$

We obtain

$$\int_{Ц} v v_t \eta^2 \le -\frac{1}{4}\int_{Ц} |\nabla(v\eta)|^2 + \frac{5}{2}\int_{Ц} |\nabla\eta|^2 v^2.$$

The left side here is equal to

$$\frac{1}{2}\int_{Ц} (v^2)_t \eta^2 = \frac{1}{2}\int_{B_R^z} v^2\eta^2\Big|_0^T - \int_{Ц} v^2 \eta_t \eta,$$

and since $\eta|_{t=0} = 0$, from the last two inequalities we obtain (3.11).

Lemma 3.2. *Suppose, under the conditions of Theorem* 3.1, *that* $\widetilde{Ц} = B_{R_1}^z \times (T_1, T)$, *where* $0 < T_1 < T$, $0 < R_1 < R$. *Set*

$$H = \int_{Ц} u_+^2, \qquad \widetilde{H} = \int_{\widetilde{Ц}} (u-\theta)_+^2, \quad \theta > 0.$$

Then

$$\widetilde{H} \le \frac{CH}{\delta\Lambda(C\delta^{-1}\theta^{-2}H)}, \tag{3.13}$$

where $\delta = \min(T_1, (R-R_1)^2)$ *and* $C > 0$ *is an absolute constant.*

Proof. In (3.11) we set $\eta(x, t) = \eta_1(x)\eta_2(t)$, where the function $\eta_1(x)$ is equal to 1 in the ball $B_{(R+R_1))/2}^z$ and to zero outside the ball B_R^z, and is linear along the radius in the layer $B_R^z \setminus B_{(R+R_1)/2}^z$; the function $\eta_2(t)$ is equal to 1 for $t \ge T_1$ and equal to t/T_1 for $t < T_1$. In (3.11) we also set $v = u_+$, and in place of T we take an arbitrary time $\tau \in [T_1, T]$. We obtain

$$\int_{B_{(R+R_1)/2}^z} u(x, \tau)_+^2\, dx \le 5\int_{Ц} u_+^2(|\nabla\eta|^2 + |\eta\eta_t|) \le \frac{25H}{\delta}. \tag{3.14}$$

We have here used the fact that $|\nabla\eta|^2 \le 4/(R-R_1)^2 \le 4/\delta$ and $|\eta\eta_t| \le 1/T_1 \le 1/\delta$.

We now apply (3.11) to the function $v = (u-\theta)_+$ (where $\theta > 0$); we set $\eta_1(x)$ equal to 1 in $B_{R_1}^z$ and to zero outside $B_{(R+R_1)/2}^z$, and take it to be linear between these two balls, while the function η_2 remains as before. We then obtain

$$\int_{Ц} |\nabla(v\eta)|^2 \le \frac{50}{\delta}\int_{Ц} v^2. \tag{3.15}$$

Since for each fixed t the function $v\eta$ has support in $\overline{D}_t$, where $D_t = \{x \in B_{(R+R_1)/2}^z : u(x, t) > \theta\}$, according to the variational property of the first eigenvalue,

$$\int_{B_R^z} |\nabla(v\eta)|^2 \ge \lambda_1(D_t)\int_{B_R^z} (v\eta)^2. \tag{3.16}$$

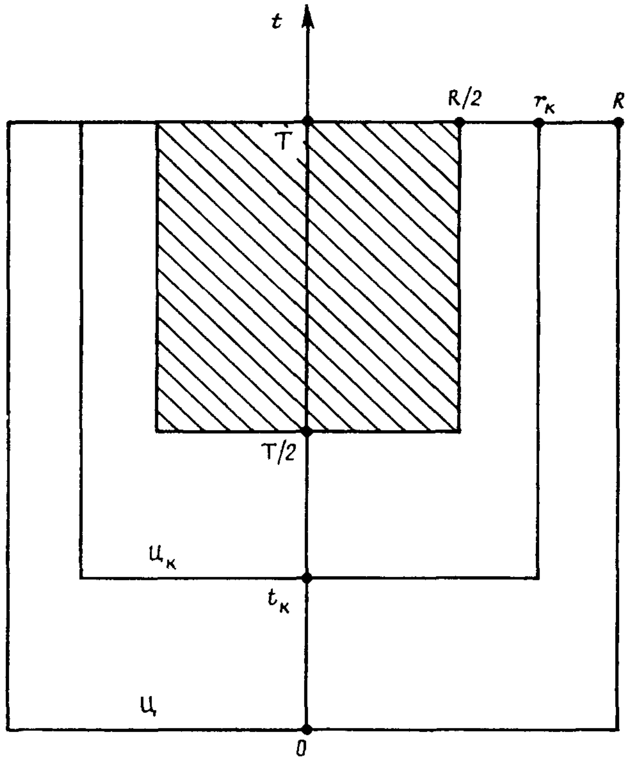

FIGURE 4

For $t \in [T_1, T]$ it follows from (3.14) that

(3.17) $$\operatorname{meas}_n D_t \le \frac{1}{\theta^2} \int_{B^z_{(R+R_1)/2}} u_+^2(x, t)\, dx \le \frac{25H}{\theta^2 \delta}.$$

Using the isoperimetric inequality (3.1), from (3.15)–(3.17) we obtain

$$\frac{50}{\delta} \int_{Ц} v^2 \ge \Lambda(25H\theta^{-2}\delta^{-1}) \int_{T_1}^{T} \int_{B^z_R} (v\eta)^2 \ge \Lambda(25H\theta^{-2}\delta^{-1})\widetilde{H}.$$

Finally, noting that $\int_{Ц} v^2 \le H$, we obtain (3.13) for $C = 50$.

We are now ready to proceed directly to the proof of Theorem 3.1. We consider a sequence (Figure 4) of imbedded cylinders $Ц_k = B^z_{r_k} \times (t_k, T)$, $k = 0, 1, 2, \dots$, where $0 = t_0 < t_1 < t_2 < \cdots \le T/2$, $R = r_0 > r_1 > r_2 > \cdots \ge R/2$, and, moreover, $(r_k - r_{k+1})^2 = t_{k+1} - t_k \equiv \delta_k$. Let $\theta > 0$. We set $\theta_k = (2 - 2^{-k})\theta$ and $H_k = \int_{Ц_k} (u - \theta_k)_+^2$. Obviously, the sequence $\{H_k\}$ decreases monotonically. We shall find θ for which H_k tends to 0 as $k \to \infty$. Obviously, then,

$$\int_{B^z_{R/2}} \int_{T/2}^{T} (u - 2\theta)_+^2 = 0,$$

so that $u(z, T) \le 2\theta$, i.e.

(3.18) $$u(z, T)_+^2 \le 4\theta^2,$$

whence (3.8) follows (for suitable θ).

Using Lemma 3.2 for the cylinders $Ц_k \supset Ц_{k+1}$ and the functions $u - \theta_k$ and $u - \theta_{k+1}$, we obtain

$$H_{k+1} \le \frac{CH_k}{\delta_k \Lambda(C\delta_k^{-1}4^{k+1}\theta^{-2}H_k)}. \tag{3.19}$$

We shall show that for a suitable choice of the number θ and sequence $\{\delta_k\}$ it is possible to arrange that for all $k = 0, 1, 2, \ldots$

$$H_k \le H/16^k, \tag{3.20}$$

where $H = \int_Ц u_+^2$. For $k = 0$, (3.20) is obviously satisfied. Suppose for some $k = m$ that the numbers $\delta_1, \ldots, \delta_{m-1}$ have already been chosen so that (3.20) is satisfied for $k \le m$. We shall choose δ_m so that

$$\frac{C}{\delta_m \Lambda(C\delta_m^{-1}4^{-m+1}\theta^{-2}H)} = \frac{1}{16}. \tag{3.21}$$

If we have succeeded in doing this, then from (3.19) and (3.21) we obtain $H_{m+1} \le H_m/16 \le H/16^{m+1}$. We shall show that there actually exists δ_m for which (3.21) is satisfied. We transform this equation to the form

$$\frac{C\delta_m^{-1}4^{-m+1}\theta^{-2}H}{\Lambda(C\delta_m^{-1}4^{-m+1}\theta^{-2}H)} = \frac{4^{-m+1}\theta^{-2}H}{16},$$

whence, using the definition of the function ω given at the beginning of this section, we obtain

$$\begin{gathered} C\delta_m^{-1}4^{-m+1}\theta^{-2}H = \omega(4^{-m-1}\theta^{-2}H), \\ \delta_m = 16C(4^{-m-1}\theta^{-2}H)/\omega(4^{-m-1}\theta^{-2}H). \end{gathered} \tag{3.22}$$

We now choose θ so that for all $k = 0, 1, 2, \ldots$ the inequalities $t_k \le T/2$ and $r_k \ge R/2$ are satisfied, or, equivalently,

$$\sum_{k=0}^{\infty} \delta_k \le T/2, \qquad \sum_{k=0}^{\infty} \sqrt{\delta_k} \le R/2. \tag{3.23}$$

From (3.22) it follows that

$$\sum_{k=0}^{\infty} \delta_k \le \sum_{k=1}^{\infty} \frac{16C(4^{-k}\theta^{-2}H)}{\omega(4^{-k}\theta^{-2}H)} \le 16C \int_0^{\infty} \frac{4^{-k}\theta^{-2}H\,dk}{\omega(4^{-k}\theta^{-2}H)}.$$

Making the change $\xi = 4^{-k}\theta^{-2}H$ in the integral and noting that $\ln 4 > 1$, we obtain

$$\sum_{k=0}^{\infty} \delta_k \le 16C \int_0^{\theta^{-2}H} \frac{d\xi}{\omega(\xi)}.$$

In exactly the same way

$$\sum_{k=0}^{\infty} \sqrt{\delta_k} \le \sum_{k=1}^{\infty} \frac{4\sqrt{C4^{-k}\theta^{-2}H}}{\sqrt{\omega(4^{-k}\theta^{-2}H)}} \le 4\sqrt{C} \int_0^{\theta^{-2}H} \frac{d\xi}{\sqrt{\xi\omega(\xi)}}.$$

Thus, conditions (3.23) are clearly satisfied if

$$\int_0^{\theta^{-2}H} \frac{d\xi}{\omega(\xi)} \le cT, \qquad \int_0^{\theta^{-2}H} \frac{d\xi}{\sqrt{\xi\omega(\xi)}} \le cR$$

(where $c \le 1/32C$, for example, $c = 0.0001$), i.e.,

$$\theta^{-2}H \le V(T), \qquad \theta^{-2}H \le W(R).$$

We set

$$\theta^2 = \frac{H}{\min(V(T), W(R))},$$

from (3.18) we then obtain (3.8). Theorem 3.1 is proved.

§4. Harnack's inequality

We fix a point $z \in M$ and introduce the abbreviated notation

$$B_R \equiv B_R^z, \qquad Ц_R \equiv B_R \times (0, R^2).$$

Theorem 4.1. *Suppose conditions* (a) *and* (b′) *of* §1 *are satisfied on the manifold* M. *Let* $u(x, t)$ *be a positive solution of the heat equation in* $Ц_{8R}$ *which is smooth in* $\overline{Ц}_{8R}$ *and satisfies the Neumann condition for* $x \in \partial M$ (*if* ∂M *is nonempty*). *Set* $\widetilde{Ц} = B_R \times (3R^2, 4R^2)$, *and suppose that* $\sup_{\widetilde{Ц}} u = 1$. *Then* $u(z, 64R^2) \ge \gamma$, *where* $\gamma = \gamma(A, a, N) > 0$.

The scheme of proof of this theorem is close to Landis' scheme [13].

We shall first prove a number of lemmas, assuming everywhere that conditions (a) and (b′) are satisfied. All solutions of the heat equation are assumed to satisfy the Neumann condition on ∂M.

Lemma 4.1. *Let* u *be a positive solution of the heat equation in* $Ц_{2R}$ *which is smooth in* $\overline{Ц}_{2R}$, *and set*

$$H = \{(x, t) \in Ц_R : u(x, t) > 1\}, \qquad \widetilde{Ц}_R = B_R \times (3R^2, 4R^2).$$

Then for any $\delta > 0$ *there exists* $\varepsilon = \varepsilon(\delta, A, a, N) > 0$ *such that if*

$$|H| \ge \delta |Ц_R|, \tag{4.1}$$

then $\inf_{\widetilde{Ц}_R} u \ge \varepsilon$.

Proof. We set $v = \ln(1/u)$. Then, in $Ц_{2R}$, v satisfies the equation (Figure 5)

$$v_t - \Delta v = -|\nabla v|^2 \tag{4.2}$$

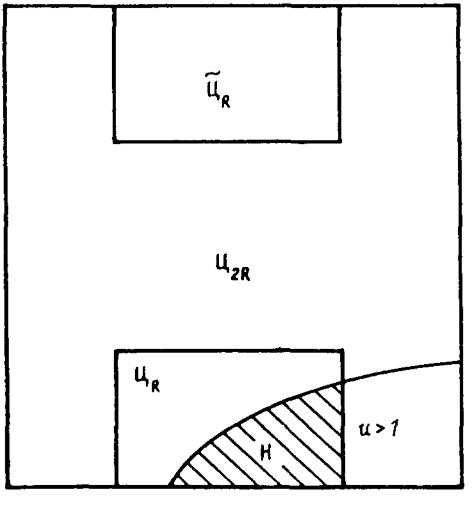

Figure 5

(and also the Neumann condition on ∂M). Moreover, obviously,

$$H = \{(x, t) \in \amalg_R : v(x, t) < 0\},$$

and it suffices to prove that $\sup_{\widetilde{\amalg}_R} v \leq C(\delta, A, a, N)$.

Let $\eta(x) \in C_0^\infty(B_{2R})$. We show that for any $t \in (0, 4R^2)$

$$\int_{B_{2R}} (v_+)_t \eta^2 \leq -\frac{1}{2} \int_{B_{2R}} |\nabla v_+|^2 \eta^2 + 2 \int_{B_{2R}} |\nabla \eta|^2. \tag{4.3}$$

Indeed, multiplying (4.2) by η^2 and integrating over the region

$$\Omega_\theta = \{(x, t) \in B_{2R} : v(x, t) > \theta\},$$

where $\theta > 0$ is a regular value of the functions $v(\cdot, t)$ and $v(\cdot, t)|_{\partial M}$, we obtain

$$\begin{aligned} \int_{\Omega_\theta} v_t \eta^2 &= \int_{\Omega_\theta} [\eta^2 \Delta v - |\nabla v|^2 \eta^2] \\ &= \int_{\partial(\Omega_\theta \setminus \partial M)} \frac{\partial v}{\partial \nu} \eta^2 - 2 \int_{\Omega_\theta} \eta(\nabla v, \nabla \eta) - \int_{\Omega_\theta} |\nabla v|^2 \eta^2, \end{aligned} \tag{4.4}$$

where ν is the outer normal with respect to Ω_θ. The boundary $\partial(\Omega_\theta \setminus \partial M)$ consists of three parts lying, respectively, on the surfaces ∂B_{2R}, $\{v = \theta\}$, and ∂M. In a neighborhood of ∂B_{2R} we have $\eta^2 = 0$, on $\{v = \theta\}$ we have $\partial v / \partial \nu \leq 0$, and on ∂M we have $\partial v / \partial \nu = 0$. Thus, the integral over $\partial(\Omega_\theta \setminus \partial M)$ in (4.4) is nonpositive. Applying the inequality $-2\eta(\nabla v, \nabla \eta) \leq \frac{1}{2}|\nabla v|^2 \eta^2 + 2|\nabla \eta|^2$ to estimate the second integral on the right side of (4.4) and letting $\theta \to 0$, we obtain (4.3). Moreover, by a limiting procedure (4.3) extends to all Lipschitz functions η with support in $\overline{B}_{2R}$. We set $\eta(x) = (d(x)/2R)^{\alpha/2}$, where $d(x)$ is the distance from the point x to ∂B_{2R}, and $\alpha = \alpha(A)$ is the constant of Theorem 1.2 (we can assume that it is sufficiently large, for example, $\alpha > 2$—we need this below). Then

$$|\nabla \eta| = \frac{\alpha}{4R} \left(\frac{d(x)}{2R} \right)^{\alpha/2 - 1} |\nabla d| \leq \frac{\alpha}{4R},$$

whence it follows that

$$\int_{B_{2R}} |\nabla \eta|^2 \leq \frac{\alpha^2}{16R^2} |B_{2R}|. \tag{4.5}$$

Let $H_t = \{(x, t) \in B_R : v(x, t) < 0\}$. According to Theorem 1.3,

$$\int_{B_{2R}} |\nabla v_+|^2 \eta^2 \geq \frac{\text{const}|H_t|}{R^2 |B_{2R}|^2} \left\{ \int_{B_{2R}} v_+ \eta^2 \right\}^2. \tag{4.6}$$

Here and below const denotes a positive constant depending only on A, a, and N. Setting $I(t) \equiv \int_{B_{2R}} v_+ \eta^2$, from (4.6), (4.5), and (4.3) we obtain

$$\frac{d}{dt} I(t) \leq -K(t) I(t)^2 + D, \tag{4.7}$$

where $K(t) = \text{const}|H_t| / R^2 |B_{2R}|^2$ and $D = \alpha^2 |B_{2R}| / 8R^2$.

We deduce from (4.7) that for all $t \in [R^2, 4R^2]$

$$I(t) \leq \left\{ \int_0^{R^2} K(\tau)\, d\tau \right\}^{-1} + Dt. \tag{4.8}$$

If $I(t^*) \leq Dt^*$ for some $t = t^* \leq R^2$, then for $t \geq t^*$ because $(d/dt)I \leq D$ we obtain $I(t) \leq Dt$ and hence also (4.8). Suppose $I(t) > Dt$ for all $t \leq R^2$. We set $J(t) = I(t) - Dt$. From (4.7) it then follows that $(d/dt)J \leq -KJ^2$. Dividing this inequality by J^2 (here we observe that $J > 0$) and integrating from 0 to R^2, we obtain

$$J(R^2) \leq \left\{ \int_0^{R^2} K(\tau)\, d\tau \right\}^{-1}.$$

Since for $t > R^2$ we have $I(t) \leq I(R^2) + D(t - R^2) = J(R^2) + Dt$, from this and the preceding estimate for $J(R^2)$ we obtain (4.8).

Substituting into (4.8) the values of $K(\tau)$ and D and noting that $t \leq 4R$, $\delta \leq 1$, and

$$\int_0^{R^2} |H_\tau|\, d\tau = |H| \geq \delta |\amalg_R| = \delta R^2 |B_R| \geq \delta R^2 A^{-1} |B_{2R}|,$$

we obtain

$$I(t) \leq \mathrm{const}|B_{2R}|/\delta \tag{4.9}$$

for all $t \in [R^2, 4R^2]$.

We integrate (4.3) with respect to t from R^2 to $4R^2$:

$$\int_{B_{2R}} v_+ \eta^2 \Big|_{R^2}^{4R^2} \leq \frac{1}{2} \int_{R^2}^{4R^2} \int_{B_{2R}} |\nabla v_+|^2 \eta^2 + \mathrm{const}|B_{2R}|.$$

Since $\eta^2|_{B_{5R/3}} \geq 1/6^\alpha$, from this and (4.9) we obtain

$$\int_{R^2}^{4R^2} \int_{B_{5R/3}} |\nabla v_+|^2 \leq 2 \cdot 6^\alpha \int_{B_{2R}} v_+ \eta^2 \big|_{t=R^2} + \mathrm{const}|B_{2R}| \leq \frac{\mathrm{const}}{\delta} |B_{2R}|. \tag{4.10}$$

We set

$$\overline{v}(t) \equiv |B_{4R/3}|^{-1} \int_{B_{4R/3}} v_+(x, t).$$

From (4.9) it follows that $\overline{v}(t) \leq \mathrm{const}/\delta$ for $t \in [R^2, 4R^2]$. Applying Theorem 1.2 (inequality (1.5)), we have

$$\int_{B_{4R/3}} v_+^2 \leq 2 \int_{B_{4R/3}} \overline{v}^2 + 2 \int_{B_{4R/3}} (v_+ - \overline{v})^2 \leq 2\overline{v}^2 |B_{4R/3}| + \mathrm{const}\, R^2 \int_{B_{5R/3}} |\nabla v_+|^2.$$

Integrating this inequality with respect to t from R^2 to $4R^2$ and using the estimate for $\overline{v}$ and (4.10), we obtain

$$\int_{R^2}^{4R^2} \int_{B_{4R/3}} v_+^2 \leq \frac{\mathrm{const}}{\delta^2} R^2 |B_{2R}|. \tag{4.11}$$

Finally, we apply Theorem 3.1 to the function v. For this we observe that by Theorem 1.4 the estimate (1.19) holds for the first eigenvalue of the Dirichlet problem, or, in terms of §3, in each ball an isoperimetric inequality with the function Λ of (3.4) holds. Then by Theorem 3.1 the estimate (3.10) holds (see Example 3 of §3). Applying it to the function v (satisfying the inequality $v_t - \Delta v \leq 0$) in the cylinder $B^x_{R/3} \times (t - (R/3)^2, t) \subset B_{4R/3} \times (R^2, 4R^2)$ (where $(x, t) \in \widetilde{\amalg}_R$), we obtain

$$v_+(x, t)^2 \leq \frac{\mathrm{const}}{R^2 |B^x_{R/3}|} \int_{t-R^2/9}^{t} \int_{B^x_{R/3}} v_+^2 \leq \frac{\mathrm{const}}{R^2 |B_{2R}|} \int_{R^2}^{4R^2} \int_{B_{4R/3}} v_+^2.$$

By (4.11) it now follows that $v(x, t) \leq \text{const}/\delta$, which was to be proved.

Lemma 4.1′. *Let u be a positive solution of the heat equation in $Ц_{2R}$ which is smooth in $\overline{Ц}_{2R}$, and set*

$$H = \{(x, t) \in Ц_R : u(x, t) < 0\}.$$

For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, A, a, N) > 0$ such that if $|H| \geq \delta |Ц_R|$, then

$$\sup_{Ц_{2R}} u \geq (1 + \varepsilon) u(z, 4R^2). \tag{4.12}$$

Proof. If $u(z, 4R^2) \leq 0$, then (4.12) is obvious. Otherwise we set $v = 1 - u/\sup u$. Obviously, $v \geq 0$, and

$$H = \{(x, t) \in Ц_R : v(x, t) > 1\}.$$

By Lemma 4.1, $|H| \geq \delta |Ц_R|$ implies $\inf_{\widetilde{Ц}_R} v \geq \varepsilon$, $v(z, 4R^2) \geq \varepsilon$, and $u(z, 4R^2)/\sup u \leq 1 - \varepsilon$, whence (4.12) follows.

Lemma 4.2. *Let u be a positive solution of the heat equation in $Ц_{2R}$ which is smooth in $\overline{Ц}_{2R}$. Let $Ц^0 \equiv B_r^y \times (\tau, \tau + r^2) \subset Ц_{R/2}$. Let $H = \{(x, t) \in Ц^0 : u(x, t) > 1\}$. If $|H| \geq \delta |Ц^0|$, then*

$$u(z, 4R^2) \geq c(|Ц^0|/|Ц_R|)^l,$$

where $\delta > 0$ is arbitrary, the positive numbers c and l depend on A, a, and N, and c further depends on δ.

Proof. We consider the cylinders (Figure 6)

$$Ц^k \equiv B^y_{2^k r} \times (\tau, \tau + 4^k r^2), \qquad \widetilde{Ц}^k \equiv B^y_{2^k r} \times (\tau + 3 \cdot 4^k r^2, \tau + 4^{k+1} r^2),$$

where $k = 0, 1, 2, \dots$. Applying Lemma 4.1 to the solution u in the cylinders $Ц^1$, $Ц^0$, and $\widetilde{Ц}^0$, we obtain $\inf_{\widetilde{Ц}^0} u \geq \varepsilon_1 \equiv \varepsilon(\delta, A, a, N)$. We consider the function u/ε_1 in the cylinders $Ц^2$, $Ц^1$, and $\widetilde{Ц}^1$. Since $u/\varepsilon_1|_{\widetilde{Ц}^0} \geq 1$ and $|Ц^0| \geq (4A)^{-1} |Ц^1|$, by Lemma 4.1

$$\inf_{\widetilde{Ц}^1} u/\varepsilon_1 \geq \varepsilon_2 \equiv \varepsilon((4A)^{-1}, A, a, N).$$

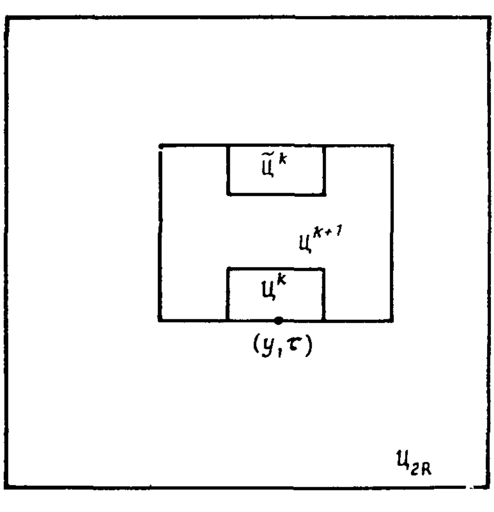

FIGURE 6

Considering the function $u/\varepsilon_1\varepsilon_2$ in the cylinders $Ц^3$, $Ц^2$, and $\widetilde{Ц}^2$, we similarly obtain $\inf_{\widetilde{Ц}^2} u/\varepsilon_1\varepsilon_2 \geq \varepsilon_2$, etc. By induction we obtain

$$\inf_{\widetilde{Ц}^k} u \geq \varepsilon_1 \varepsilon_2^k.$$

Let $\overline{k}$ be the largest index for which $Ц^{\overline{k}} \subset Ц_R$. The estimate we have obtained is then satisfied also for $k = \overline{k}$. It is not hard to see that $2^{\overline{k}+1} r \geq R/2$, and hence

$$|Ц^{\overline{k}}| = 4^{\overline{k}} r^2 |B^y_{2^{\overline{k}}r}| \geq \frac{R^2}{16} |B^y_{R/4}| \geq 1/(16A^3)|Ц_R|.$$

Applying again Lemma 4.1 to the function $u/\varepsilon_1\varepsilon_2^{\overline{k}}$ in the cylinders $Ц_{2R}$, $Ц_R$, and $\widetilde{Ц}_R$, we obtain $\inf_{\widetilde{Ц}_R} u > \varepsilon_1 \varepsilon_2^{\overline{k}} \varepsilon_3$, where $\varepsilon_3 \equiv \varepsilon((16A)^{-3}, A, a, N)$. Since $|Ц^0| 4^{\overline{k}} \leq |Ц^{\overline{k}}| \leq |Ц_R|$, it follows that $\overline{k} \leq \log_4(|Ц_R|/|Ц^0|)$. Hence,

$$\inf_{\widetilde{Ц}_R} u \geq \varepsilon_1 \varepsilon_3 (|Ц^0|/|Ц_R|)^{-\log_4 \varepsilon_2},$$

whence the required inequality follows.

Lemma 4.3. *Let u be a positive solution of the heat equation in $Ц_R$ which is smooth in $\overline{Ц}_R$. Let $E = \{(x, t) \in Ц_R : u(x, t) > 1\}$. Let $u(z, R^2) \geq 2$. Then there exists $\eta(A, a, N) > 0$ such that $|E| \leq \eta |Ц_R|$ implies $\sup_{Ц_R} u \geq 4$.*

Proof. In Lemma 4.1 we set $\delta = 1/2$ and fix $\varepsilon = \varepsilon(1/2, A, a, N)$. We shall find an integer m such that $(1+\varepsilon)^m > 3$. We set $r = R/2m$ and consider the function $u-1$ in the cylinders $Ц^0_{2r} = B_{2r} \times (R^2 - 4r^2, R^2)$ and $Ц^0_r = B_r \times (R^2 - 4r^2, R^2 - 3r^2)$. We choose η so small that $|E \cap Ц^0_r| \leq \frac{1}{2}|Ц^0_r|$. For this we note that, according to Theorem 1.1,

$$|E| \leq \eta |Ц_R| \leq \eta \frac{R^2}{r^2} A_1 \left(\frac{R}{r}\right)^{\alpha_1} |Ц^0_r| = A_1 \eta (2m)^{2+\alpha_1} |Ц^0_r|,$$

so that it is possible to set $\eta = \frac{1}{2} A_1^{-1} (2m)^{-2-\alpha_1}$. Applying Lemma 4.1′ to the function $u - 1$, we obtain

$$\sup_{Ц^0_{2r}} (u-1) \geq (1+\varepsilon)(u(z, R^2) - 1) \geq (1+\varepsilon).$$

We shall find a point $(x_1, t_1) \in \overline{Ц}^0_{2r}$ at which $u - 1 \geq 1 + \varepsilon$. Applying Lemma 4.1′ to the function $u - 1$ in the cylinders (Figure 7)

$$Ц^1_{2r} = B^{x_1}_{2r} \times (t_1 - 4r^2, t_1), \qquad Ц^1_r = B^{x_1}_r \times (t_1 - 4r^2, t_1 - 3r^2),$$

we obtain

$$\sup_{Ц^1_{2r}} (u-1) \geq (1+\varepsilon)(u(x_1, t_1) - 1) \geq (1+\varepsilon)^2,$$

since as above we have $|E \cap Ц^1_r| \leq \frac{1}{2}|Ц^1_r|$. We further find a point $(x_2, t_2) \in \overline{Ц}^1_{2r}$ at which $u - 1 \geq (1+\varepsilon)^2$, etc. We obtain a sequence of points (x_k, t_k) at which $u(x_k, t_k) - 1 \geq (1+\varepsilon)^k$, and

$$d(x_k, x_{k+1}) \leq 2r, \qquad 0 \leq t_k - t_{k+1} \leq 4r^2.$$

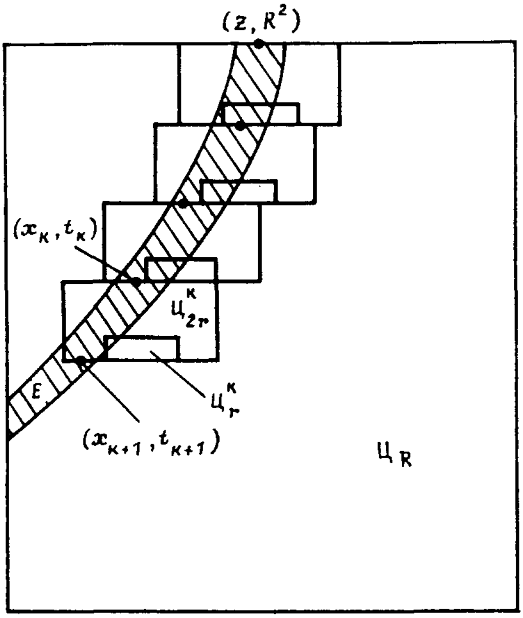

FIGURE 7

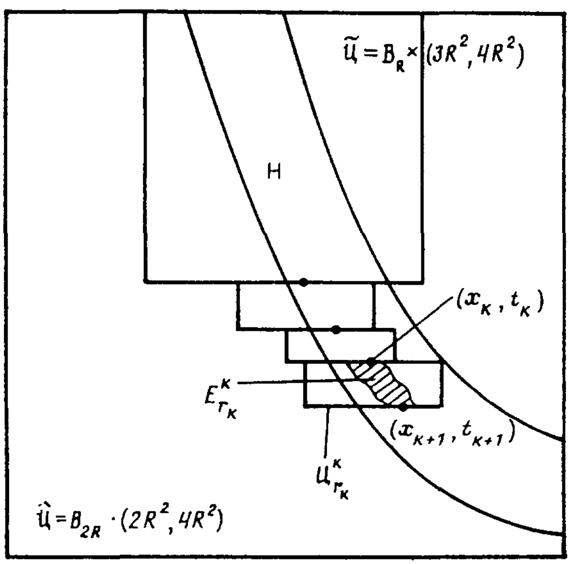

FIGURE 8

Since $4r^2 m \le R^2$ and $2rm = R$, we can construct at least m such points. By the choice of m we obtain $\sup_{\text{Ц}_R} u \ge (1+\varepsilon)^m + 1 > 3 + 1 = 4$. The lemma is proved.

We proceed directly to the proof of Theorem 4.1. Let

$$H = \{(x, t) \in \text{Ц}_{4R} \colon u(x, t) > \tfrac{1}{2}\}.$$

If $|H| \ge \delta |\text{Ц}_{4R}|$ (where $\delta > 0$, depending only on A, a, and N, will be chosen later), then, by Lemma 4.1, $u(z, 64R^2) \ge \frac{1}{2}\varepsilon(\delta, A, a, N)$, and everything is proved.

Suppose now that $|H| \le \delta |\text{Ц}_{4R}|$. Let (x_0, t_0) be a maximum point of the function u in the closure of the cylinder $\widetilde{\text{Ц}}$, i.e., $u(x_0, t_0) = 1$. For any $r > 0$ we consider the cylinder $Ц_r^0 = B_r^{x_0} \times (t_0 - r^2, t_0)$ and the set $E_r^0 = \{(x, t) \in Ц_r^0 : u(x, t) > 1/2\}$. If r is sufficiently small, then the ratio $|E_r^0|/|Ц_r^0|$ can be arbitrarily close to 1. If $r = R/2$, then $|E_r^0|/|Ц_r^0| \le \delta |Ц_{4R}|/|Ц_{R/2}^0| \le \delta \cdot 64 A_1 8^{\alpha_1}$. We set $\delta = \eta(A, a, N)/(64 A_1 8^{\alpha_1})$, where η is the constant of Lemma 4.3. Then there exists $r_0 \le R/2$ such that $|E_{r_0}^0| = \eta |Ц_{r_0}^0|$. By Lemma 4.3 we have $\sup_{Ц_{r_0}^0} u \ge 2$. Let (x_1, t_1) be a maximum point of the function u in $\overline{Ц}_{r_0}^0$; in particular, $u(x_1, t_1) \ge 2$. We consider the cylinder $Ц_r^1 = B_r^{x_1} \times (t_1 - r^2, t_1)$ and the set $E_r^1 = \{(x, t) \in Ц_r^1 : u(x, t) > 1\}$. As above, there exists $r_1 \le R/2$ such that $|E_{r_1}^1| = \eta |Ц_{r_1}^1|$, and by Lemma 4.3 we have $\sup_{Ц_{r_1}^1} u \ge 4$. Continuing this process, we find a sequence of points (x_k, t_k) for which $u(x_k, t_k) \ge 2^k$, cylinders $Ц_{r_k}^k = B_{r_k}^{x_k} \times (t_k - r_k^2, t_k)$ (where $r_k \le R/2$), and sets $E_{r_k}^k = \{(x, t) \in Ц_{r_k}^k : u(x, t) > 2^{k-1}\}$, where $|E_{r_k}^k| = \eta |Ц_{r_k}^k|$, $\sup_{Ц_{r_k}^k} u \ge 2^{k+1}$, and $(x_{k+1}, t_{k+1}) \in \overline{Ц}_{r_k}^k$ (Figure 8). We consider those k for which $Ц_{r_k}^k \subset \widehat{Ц} \equiv B_{2R} \times (2R^2, 4R^2)$. The set of such k is finite, since $\sup u < \infty$. Let $\overline{k}$ be the largest such index. Since the cylinder $Ц_{r_{\overline{k}+1}}^{\overline{k}+1}$ does not lie in $\widehat{Ц}$, it follows that $r_0 + r_1 + \cdots + r_{\overline{k}+1} \ge R$. Since $r_k \le R/2$, it follows that $r_0 + r_1 + \cdots + r_{\overline{k}} \ge R/2 > (1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots)R/4$. Therefore, there exists $k \le \overline{k}$ such that $r_k \ge (k+1)^{-2} R/4$. We fix this k and apply Lemma 4.2 to the function $u/2^{k-1}$ in the cylinders $Ц_{r_k}^k \subset Ц_{2R}$ and $Ц_{8R}$:

$$u(z, 64R^2) \ge c(|Ц_{r_k}^k|/|Ц_{4R}|)^l 2^{k-1}.$$

Since $|Ц_{r_k}^k|/|Ц_{4R}| \ge A^{-1}(r_k/R)^{\alpha_1+2} \ge \mathrm{const}(k+1)^{2\alpha_1+4}$, it follows that

$$u(z, 64R^2) \ge c \cdot \mathrm{const} \frac{2^{k-1}}{(k+1)^{(2\alpha_1+4)l}} \ge \gamma(A, a, N) \ge 0,$$

because the infimum over k of the fraction in this expression is a positive constant depending on α_1 and l, i.e., in the final analysis on A, a, and N. The theorem is proved.

§5. Necessary conditions for Harnack's inequality

Theorem 5.1. *Suppose that on a Riemannian manifold M Harnack's inequality is satisfied in the form in Theorem* 4.1 *with a constant $\gamma > 0$ not depending on R. Then*:

1) *For any $x \in M$ and $R > 0$*

$$|B_{2R}^x| \le A |B_R^x|, \tag{5.1}$$

where $A = A(\gamma)$.

2) *For any ball B_R^y and any domain Ω, $\overline{\Omega} \subset B_R^y$,*

$$\lambda_1(\Omega) \ge \frac{b}{R^2} \left\{ \frac{|B_R^y|}{|\Omega|} \right\}^\beta, \tag{5.2}$$

where b, $\beta > 0$ depend on γ.

For the proof of Theorem 5.1 we use the Green function $G(x, y, t)$ of the heat equation. By definition, this is the smallest positive fundamental solution of the heat equation (with the Neumann condition on ∂M if the boundary is nonempty). It is known (see [1]) that the Green function exists on any manifold. It can be constructed as the limit $\Omega \to M$ (we thus denote the exhaustion of the manifold

M by precompact open sets Ω) of the Green functions $G_\Omega(x, y, t)$ of the mixed problem

$$u_t - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{5.3}$$

$$u|_{t=0} = f, \tag{5.4}$$

$$u|_{\partial\Omega} = 0, \qquad \frac{\partial u}{\partial \nu}\Big|_{\partial M \cap \Omega} = 0 \tag{5.5}$$

(where f is the initial function; we assume that $f \in C_0^\infty(\Omega)$). In the case where the domain Ω has a smooth boundary $\partial\Omega$ transversal to ∂M (and we may assume that this is so), the Green function of this problem exists in the classical sense and has the following properties:

(a) $G_\Omega(x, y, t) \geq 0$, $G_\Omega(x, y, t) = G_\Omega(y, x, t)$, and $G_\Omega(x, y, t)$ satisfies equation (5.3) both in x and y, along with the boundary conditions (5.5).

(b) The function

$$u(x, t) = \int_\Omega G_\Omega(x, y, t) f(y)\, dy$$

is the solution of problem (5.3)–(5.5);

(c) If φ_k and λ_k are the kth eigenfunction and corresponding eigenvalue of the Dirichlet problem (1.18) in the domain Ω, and the functions φ_k have unit $L_2(\Omega)$-norm, then for any $x, y \in \Omega$ and $t > 0$

$$G_\Omega(x, y, t) = \sum_{k=1}^{\infty} \exp(-\lambda_k t)\varphi_k(x)\varphi_k(y). \tag{5.6}$$

From the maximum principle it follows, first of all, that

$$\int_\Omega G_\Omega(x, y, t)\, dy \leq 1,$$

and, second, on expansion of Ω the function $G_\Omega(x, y, t)$ increases. Hence the limit

$$G(x, y, t) = \lim_{\Omega \to M} G_\Omega(x, y, t),$$

which is the Green function of the heat equation on M, exists. It is important for us that $G(x, y, t)$ is positive, symmetric in x and y, and satisfies the heat equation (and also the Neumann condition on a nonempty boundary ∂M) so that for any locally summable function $f \geq 0$ the formula

$$u(x, t) = \int_M G(x, y, t) f(y)\, dy \tag{5.7}$$

defines the smallest positive solution of the Cauchy problem

$$u_t - \Delta u = 0 \quad \text{in } M \times (0, +\infty),$$
$$u|_{t=0} = f, \qquad \frac{\partial u}{\partial \nu}\Big|_{\partial M} = 0$$

(more precisely, a positive solution of the Cauchy problem exists if and only if the integral in (5.7) converges, and in this case the smallest positive solution is given by (5.7)), and, finally,

$$\int_M G(x, y, t)\, dy \leq 1. \tag{5.8}$$

The reader can find a detailed justification of this construction in [1].

Proceeding directly to the proof of the theorem, we use the following lemma.

Lemma 5.1. *Under the conditions of Theorem* 5.1, *for any positive solution* $u(x, t)$ *of the heat equation in* $M \times (0, +\infty)$, *for any* $x, y \in M$ *and* $\tau > t > 0$,

$$u(x, t) \le u(y, \tau) \exp\left\{C\left(\frac{\tau}{t} + \frac{r^2}{\tau - t}\right)\right\},$$

where $r = d(x, y)$ *and* $C = C(\gamma) > 0$.

The proof of this lemma (under the condition that Harnack's inequality is satisfied) is well known; see [5], [6], and [10].

1) We fix a point $z \in M$ and set $B_R^z \equiv B_R$ and $u(x, t) \equiv G(x, z, t)$. Since for any $\tau > 0$

$$\int_M u(x, \tau)\, dx \le 1,$$

for any $R > 0$ there exists a point $y \in B_R$ such that

$$u(y, \tau) \le 1/|B_R|.$$

We set $\tau = 2t$ and $R = 2\sqrt{t}$, and by Lemma 5.1 we obtain

$$u(z, t) \le |B_{2\sqrt{t}}|^{-1} \exp(6C). \tag{5.9}$$

We seek a lower bound for $u(z, t)$. We first prove that for all $x \in M$ and $t > 0$

$$\int_{B_{\sqrt{t}}} u(x, t/2)\, dx \ge \gamma.$$

For this we consider the function

$$w(y, \tau) \equiv \int_{B_{\sqrt{t}}} G(x, y, \tau)\, dx$$

and continue it by one for $\tau \le 0$. Then $w(y, \tau)$ is a positive solution of the heat equation in the cylinder $B_{\sqrt{t}} \times (-\infty, +\infty)$. Applying Harnack's inequality to this function in the cylinder $B_{\sqrt{t}} \times (-t/2, t/2)$, we obtain

$$\int_{B_{\sqrt{t}}} u(x, t/2)\, dx = w(z, t/2) \ge \gamma.$$

Hence, there exists a point $x \in B_{\sqrt{t}}$ such that

$$u(x, t/2) \ge \gamma/|B_{\sqrt{t}}|.$$

Applying again Lemma 5.1, we obtain

$$u(x, t/2) \le u(z, t) \exp(4C).$$

Comparing with (5.9), we get

$$|B_{2\sqrt{t}}| \le \gamma^{-1} \exp(10C) |B_{\sqrt{t}}|.$$

Redenoting $\sqrt{t}$ by R, we obtain the desired result.

2) We rewrite (5.9) in other notation, setting $z = x$:

$$G(x, x, t) \le c/|B^x_{\sqrt{t}}|,$$

where $c = \exp(6C)$. Since $G_\Omega \le G$, for $x \in \Omega$ for all $t > 0$ we have

$$G_\Omega(x, x, t) \le c/|B^x_{\sqrt{t}}|.$$

Integrating (5.6) for $x = y$ over Ω and noting that $\int_\Omega \varphi_k^2 = 1$, we obtain

$$\int_\Omega G_\Omega(x, x, t)\,dx = \sum_{k=1}^{\infty} \exp(-\lambda_k t).$$

Hence,

$$c\int_\Omega \frac{dx}{|B^x_{\sqrt{t}}|} \ge \exp(-\lambda_1 t). \tag{5.10}$$

By hypothesis, $\overline{\Omega} \subset B^y_R$. By Theorem 1.1, (whose condition is satisfied according to part 1) of the present theorem, we have

$$|B^x_{\sqrt{t}}| \ge A_1^{-1}(\sqrt{t}/R)^{\alpha_1}|B^y_R|$$

for $\sqrt{t} \le R$ and

$$|B^x_{\sqrt{t}}| \ge A_2(\sqrt{t}/R)^{\alpha_2}|B^y_R|$$

for $\sqrt{t} > R$ (where $A_{1,2}, \alpha_{1,2} > 0$ depend on γ).

These two inequalities can be combined in a single relation valid for all $t > 0$:

$$|B^x_{\sqrt{t}}| \ge A_3 f(\sqrt{t}/R)|B^y_R|,$$

where $A_3 = \min(A_1^{-1}, A_2)$

$$f(\xi) = \begin{cases} \xi^{\alpha_1}, & \xi \le 1, \\ \xi^{\alpha_2}, & \xi > 1. \end{cases}$$

Substituting this in (5.10), we obtain

$$\frac{A_3^{-1}c|\Omega|}{|B^y_R|f(\sqrt{t}/R)} \ge \exp(-\lambda_1 t), \qquad \lambda_1 \ge t^{-1}\ln\left(A_3 c^{-1}\frac{|B^y_R|}{|\Omega|}f(\sqrt{t}/R)\right). \tag{5.11}$$

We choose t from the condition

$$f(\sqrt{t}/R) = 2A_3^{-1}c\frac{|\Omega|}{|B^y_R|}.$$

Then

$$\frac{\sqrt{t}}{R} = f^{-1}\left(2A_3^{-1}c\frac{|\Omega|}{|B^y_R|}\right) \le (1 + 2A_3^{-1}c)^{1/\alpha_2}\left(\frac{|\Omega|}{|B^y_R|}\right)^{1/\alpha_1}$$

(here we have used the following property of the function f: if $a > 1$ and $b < 1$, then $f(ab) > a^{\alpha_2}b^{\alpha_1}$; it is obviously satisfied for $\alpha_1 > \alpha_2$, and we may always consider α_1 arbitrarily large and α_2 arbitrarily close to 0). Substituting the value of t into (5.11), we obtain (5.2).

Conclusion

As we know, Harnack's inequality for the heat equation implies two-sided estimates of the Green function (see, for example, [10] and [6]). These estimates have the form

$$\frac{C_2}{|B^x_{\sqrt{t}}|}\exp\left(-c_2\frac{r^2}{t}\right) \le G(x, y, t) \le \frac{C_1}{|B^x_{\sqrt{t}}|}\exp\left(-c_1\frac{r^2}{t}\right), \tag{5.12}$$

where $r = d(x, y)$, $c_1 > 0$ may be any number $< 1/4$, $C_{1,2}, c_2 > 0$ depend on the constant γ in Harnack's inequality, and C_1 depends also on c_1. For a certain class of unbounded domains in $\mathbf{R}^n$ satisfying conditions close to (a) and (b) an analogous upper bound for the Green function was obtained by Gushchin and his coauthors in [14] and [15].

From (5.12) by integration with respect to t we obtain the following estimate of the Green function $g(x, y)$ (i.e., the least positive fundamental solution) for the Laplace equation:

$$C_2\int_r^\infty \frac{\xi\, d\xi}{|B^x_\xi|} \le g(x, y) \le C_1\int_r^\infty \frac{\xi\, d\xi}{|B^x_\xi|}, \tag{5.13}$$

where $C_{1,2} > 0$ depend on γ.

For manifolds of nonnegative Ricci curvature the estimates (5.13) were first obtained by Varopoulos [11] by elliptic methods. The parabolic estimates (5.12) on these same manifolds were obtained by Li and Yau [6]. Since, as shown in §2, on manifolds of nonnegative Ricci curvature conditions (a) and (b′) are satisfied, all these estimates follow from our results. Moreover, it is easy to see that conditions (a) and (b′) are invariant relative to quasi-isoperimetric transformations (i.e., diffeomorphisms of the manifold M changing distances by no more than a constant). From this it follows that if these conditions are satisfied then Harnack's inequality (and with it (5.12) and (5.13)) holds for solutions of the equation

$$u_t - Lu = 0,$$

where L is a uniformly elliptic operator on M going over into the Laplacian under a quasi-isometry. By the way, it is possible from the very beginning to consider the still more general parabolic equation

$$p(x)u_t - \operatorname{div}(A(x, t)\nabla u) = 0,$$

where $A(x, t)$ is a linear operator in T_xM. All our proofs (except those in §5) go through also for this equation.

It would be interesting to see if condition (b′) is necessary for Harnack's inequality. We have only been able to show that the isoperimetric inequality (5.2), weaker than (b′), is necessary. It would also be interesting to know (in the case of a negative answer to the preceding question) if Harnack's inequality is preserved under quasi-isometric transformations of the manifold.

There is an example of a manifold of dimension $n > 3$ on which condition (a) is satisfied, while condition (b′) is satisfied in a weakened form: in place of the factor $1/R^2$ in (1.2) there is $1/R^2 \ln^\gamma(R+2)$, where $\gamma > 2/(n-3)$, and Harnack's inequality for the Laplace and heat equations is not satisfied. Unfortunately, this example is too cumbersome to be presented here.

We further note a curious fact: the elliptic Harnack inequality is not only logically but actually weaker than the parabolic inequality. Indeed, as shown above, the parabolic Harnack inequality implies condition (a), while in the case of a two-dimensional manifold M for the validity of Harnack's inequality for the Laplace equation it suffices that for some $x \in M$ as $R \to \infty$ we have $|B^x_R| \le \text{const}\, R^2$ (see

[17]). It is clear that this condition does not imply (a). In this connection all the questions formulated above are of interest also for the elliptic Harnack inequality.

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Moscow

Received 7/JULY/88 and 15/AUG/90

Translated by J. R. SCHULENBERGER