Lectures on path homology and Hodge Laplacian on digraphs

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1 Spaces of ∂-invariant paths

1.1 The boundary operator on digraph paths

Let V be a finite set whose elements will be called vertices. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V (allowing repetitions). Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Any element of Λ_p is called a p-path.

An elementary p-path $i_0, ..., i_p$ as an element of Λ_p will be denoted by $e_{i_0...i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \quad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle$$

Any p-path u can be written in a form $u = \sum_{i_0, i_1, ..., i_p \in V} u^{i_0 i_1 ... i_p} e_{i_0 i_1 ... i_p}$, where $u^{i_0 i_1 ... i_p} \in \mathbb{K}$.

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\widehat{i_q}...i_p}, \tag{1.1}$$

where $\widehat{}$ means omission of the index. Set $\Lambda_{-1} = \{0\}$ and define $\partial : \Lambda_0 \to \Delta_{-1}$ by $\partial = 0$.

For example, $\partial e_i = 0$, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 1.1. *We have $\partial^2 = 0$.*

Proof. Indeed, for any $p \geq 2$ we have

$$\begin{aligned}
\partial^2 e_{i_0...i_p} &= \sum_{q=0}^{p} (-1)^q \partial e_{i_0...\widehat{i_q}...i_p} \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r e_{i_0...\widehat{i_r}...\widehat{i_q}...i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0...\widehat{i_q}...\widehat{i_r}...i_p} \right) \\
&= \sum_{0 \leq r < q \leq p} (-1)^{q+r} e_{i_0...\widehat{i_r}...\widehat{i_q}...i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} e_{i_0...\widehat{i_q}...\widehat{i_r}...i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0...i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$. ■

Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \overset{\partial}{\leftarrow} \Lambda_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \dots$$

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and irregular otherwise.

Let $\mathcal{I}_p$ be the subspace of Λ_p spanned by irregular p-paths $e_{i_0...i_p}$. We claim that $\partial \mathcal{I}_p \subset \mathcal{I}_{p-1}$. Indeed, if $e_{i_0...i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\partial e_{i_0...i_p} = e_{i_1...i_p} - e_{i_0 i_2...i_p} + ...$$

$$+ (-1)^k e_{i_0 \ldots i_{k-1} i_{k+1} i_{k+2} \ldots i_p} + (-1)^{k+1} e_{i_0 \ldots i_{k-1} i_k i_{k+2} \ldots i_p} \quad (1.2)$$
$$+ \ldots + (-1)^p e_{i_0 \ldots i_{p-1}}.$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.2) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0 \ldots i_p} \in \mathcal{I}_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / \mathcal{I}_p$, and we obtain the chain complex $\mathcal{R}_* (V)$:

$$0 \leftarrow \mathcal{R}_0 \stackrel{\partial}{\leftarrow} \mathcal{R}_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \mathcal{R}_{p-1} \stackrel{\partial}{\leftarrow} \mathcal{R}_p \stackrel{\partial}{\leftarrow} \ldots$$

By setting all irregular p-paths to be equal to 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$ in $\mathcal{R}_2$.

1.2 Spaces Ω_p of ∂-invariant paths

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and $E \subset \{V \times V \setminus \mathrm{diag}\}$ is a set of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0 \ldots i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, \ldots, p-1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p (G)$ be $\mathbb{K}$-linear subspace of Λ_p spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \langle e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed} \rangle.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{c}{\bullet}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is non-allowed.

Consider the following subspace of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p (G) := \{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \}.$$

We claim that $\partial \Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial (\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Definition. The elements of Ω_p are called ∂-*invariant* p-paths.

Thus, we obtain a *path chain complex* $\Omega_* = \Omega_* (G)$:

$$0 \leftarrow \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \ldots \quad (1.3)$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

For a vector space U over $\mathbb{K}$ we write

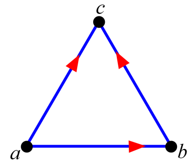
$$|U| = \dim_{\mathbb{K}} U.$$

Proposition 1.2. *If* $|\Omega_p| \leq 1$ *for some* p *then* $\Omega_n = \{0\}$ *for all* $n > p$.

1.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three distinct vertices a, b, c such that $a \to b \to c,\ a \to c$.

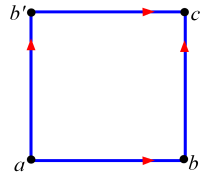
It determines a 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.



A *square* is a sequence of four distinct vertices a, b, b', c such that $a \to b \to c$, $a \to b' \to c$ while $a \not\to c$.

It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

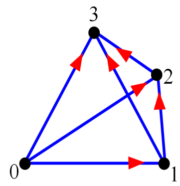
$$\begin{aligned}\partial u &= (e_{bc} - \underline{e_{ac}} + e_{ab}) - (e_{b'c} - \underline{e_{ac}} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.\end{aligned}$$



A p-*simplex* (or p-clique) is a configuration of $p+1$ distinct vertices, say, $0, 1, ..., p$, such that $i \to j\ \ \forall i < j$.

It determines a p-path $e_{01\ldots p} \in \Omega_p$.

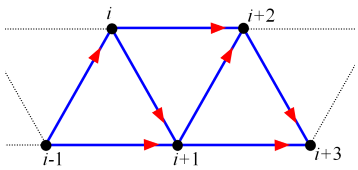
Here is a 3-simplex:



A p-*snake* is a configuration of $p+1$ distinct vertices, say $0, 1, \ldots, p$, with the following arrows:

$i \to i+1$ for all $i = 0, ..., p-1$,

$i \to i+2$ for all $i = 0, ..., p-2$.



In particular, any triple $i\,(i+1)\,(i+2)$ forms a triangle for $i = 0, \ldots, p-2$.

A p-snake determines a ∂-invariant p-path $e_{01\ldots p}$. Indeed, this path is obviously allowed, and its boundary

$$\partial e_{01\ldots p} = \sum_{q=0}^{p} (-1)^q \, e_{0\ldots(q-1)(q+1)\ldots p}$$

is also allowed because $q-1 \to q+1$. Hence, $e_{i_0\ldots i_p} \in \Omega_p$.

Clearly, a p-simplex contains a p-snake.

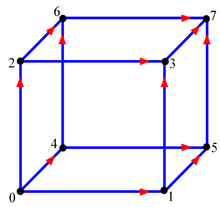
A 3-*cube* is a sequence of 8 vertices $0, 1, 2, 3, 4, 5, 6, 7$, connected by arrows as shown here:

A 3-cube determines a ∂-invariant 3-path

$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$

because $u \in \mathcal{A}_3$ and

$$\begin{aligned}\partial u = {} & (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) \\ & - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2.\end{aligned}$$



1.4 Path homology

Definition. *Path homologies* of G are defined as the homologies of the chain complex $\Omega_*(G)$:

$$H_p = H_p(G) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

The elements of $\ker \partial|_{\Omega_p}$ are called *closed* p-paths, and the elements of $\operatorname{Im} \partial|_{\Omega_{p+1}}$ are called *boundaries*.

Define the Betti numbers of G by

$$\beta_p = \beta_p(G) = |H_p|.$$

If the sequence $\{\Omega_p\}$ is finite in the sense that $\Omega_p = \{0\}$ for large enough p, then define the Euler characteristic of G by

$$\chi := \sum_{p=0}^{\infty} (-1)^p |\Omega_p| = \sum_{p=0}^{\infty} (-1)^p \beta_p.$$

Proposition 1.3. *If X and Y are two disjoint digraphs then*

$$\beta_p(X \sqcup Y) = \beta_p(X) + \beta_p(Y). \tag{1.4}$$

Proof. Clearly, any allowed elementary p-path on $X \sqcup Y$ is contained in X or Y. It follows that any ∂-invariant path on $X \sqcup Y$ is a sum of ∂-invariant paths on X and Y, that is,

$$\Omega_p(X \sqcup Y) = \Omega_p(X) \oplus \Omega_p(Y).$$

Hence, the same identity holds for homology groups, whence (1.4) follows. ∎

Proposition 1.4. *We have $\beta_0(G) = \#$of connected components of G.*

Proof. It suffices to prove that if G is connected then $\beta_0 = 1$. We have $\beta_0 = |\Omega_0| - |\partial\Omega_1|$. Let the set of vertices of G be $\{0, ..., n-1\}$ so that $|\Omega_0| = n$. Since Ω_1 is spanned by all arrows e_{ij}, $i \to j$, the space $\partial\Omega_1$ is spanned by all differences $e_j - e_i$ where $i \to j$. Since there is an edge path between the vertex 0 and any other vertex i, it follows that $\partial\Omega_1$ contains $e_i - e_0$ for any vertex $i > 0$. These $n-1$ elements of $\partial\Omega_1$ are linearly independent while $e_j - e_i = (e_j - e_0) - (e_i - e_0)$. Hence, $|\partial\Omega_1| = n - 1$ and $\beta_0 = 1$. ∎

1.5 Structure of Ω_2

As we know, $\Omega_0 = \langle e_i \rangle$ consists of linear combinations of all vertices and $\Omega_1 = \langle e_{ij} : i \to j \rangle$ consists linear combinations of all arrows.

Definition. Let us call a *semi-arrow* any pairs (x, y) of distinct vertices x, y such that $x \not\to y$ but $x \to z \to y$ for some vertex z. We write in this case $x \rightharpoonup y$

Theorem 1.5. (a) *We have $|\Omega_2| = |\mathcal{A}_2| - s$ where s is the number of semi-arrows.*

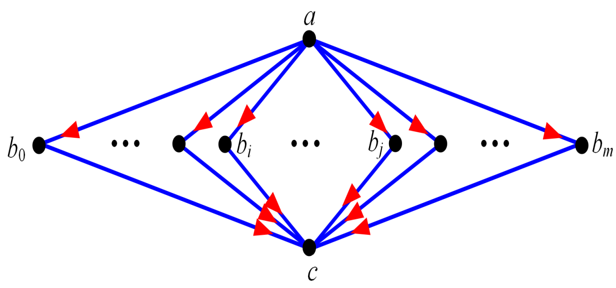
(b) *The space* Ω_2 *is spanned by all triangles* e_{abc}*, squares* $e_{abc} - e_{ab'c}$ *and double arrows* e_{aba}.

Proof. (a) Recall that
$$\mathcal{A}_2 = \operatorname{span}\{e_{abc} : abc \text{ is allowed}\}$$
and
$$\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\} = \{v \in \mathcal{A}_2 : \partial v = 0 \bmod \mathcal{A}_1\}.$$
If abc is allowed then $a \to b$ and $b \to c$, whence
$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \bmod \mathcal{A}_1.$$
If $a = c$ or $a \to c$ then $e_{ac} = 0 \bmod \mathcal{A}_1$. Otherwise ac is a semi-arrow, and in this case
$$e_{ac} \neq 0 \mod \mathcal{A}_1.$$
For any $v \in \mathcal{A}_2$, we have
$$v = \sum_{\{a \to b \to c\}} v^{abc} e_{abc} \tag{1.5}$$
from which it follows that
$$\partial v = -\sum_{\{a \to b \to c, a \rightharpoonup c\}} v^{abc} e_{ac} \mod \mathcal{A}_1.$$
The condition $\partial v = 0 \bmod \mathcal{A}_1$ is equivalent to
$$\sum_{\{a \to b \to c,\ a \rightharpoonup c\}} v^{abc} e_{ac} = 0 \bmod \mathcal{A}_1,$$
which in turn is equivalent to
$$\sum_{b : a \to b \to c} v^{abc} = 0 \text{ for any semi-arrow } ac. \tag{1.6}$$
The number of the equations in (1.6) is exactly s, and they all are linearly independent for different semi-arrows, because a triple abc determines at most one semi-arrow. Hence, Ω_2 is obtained from $\mathcal{A}_2$ by imposing s linearly independent conditions, whence $|\Omega_2| = |\mathcal{A}_2| - s$.

(b) Fix $v \in \Omega_2$ and prove that v is a linear combination of triangles, double arrows and squares. As v is allowed, it admits representation (1.5) as a linear combination of allowed elementary 2-paths e_{abc}. If $c = a$ then e_{abc} is a double arrow. If $a \neq c$ and $a \to c$ then e_{abc} is a triangle. Subtracting from v all double arrows and triangles, we can assume that v has no such terms any more. Then, for any term e_{abc} in v with a non-zero coefficient v^{abc}, we have $a \neq c$ and $a \not\to c$; that is, ac is a semi-arrow.

For any semi-arrow ac set
$$v_{ac} = \sum_{b : a \to b \to c} v^{abc} e_{abc}.$$
Since $v = \sum_{a \rightharpoonup c} v_{ac}$, it suffices to prove that each v_{ac} is a linear combination of squares.

Denote by $\{b_0, b_1, ..., b_m\}$ the sequence of all possible vertices b s.t. $a \to b \to c$. This configuration is called m*-square*: (for example, a square is 1-square).



Then we have

$$v_{ac} = \sum_{i=0}^{m} v^{ab_ic} e_{ab_ic} = \sum_{i=1}^{m} v^{ab_ic} e_{ab_ic} + v^{ab_0c} e_{ab_0c} = \sum_{i=1}^{m} v^{ab_ic} (e_{ab_ic} - e_{ab_0c})$$

because by (1.6)

$$v^{ab_0c} = -\sum_{i=1}^{m} v^{ab_ic}.$$

Hence, v_{ac} is a linear combination of squares $e_{ab_ic} - e_{ab_0c}$, which completes the proof. ∎

Example. Let the digraph G be an m-square shown on the above picture. It has one semi-arrow $a \rightharpoonup c$ so that $s = 1$. Since $|\mathcal{A}_2| = m + 1$, we conclude that $|\Omega_2| = m$. Indeed, the basis in Ω_2 is given by the sequence of m squares $\{e_{ab_ic} - e_{ab_0c}\}_{i=1}^{m}$.

1.6 Digraph morphisms

Let X and Y be two digraphs. For simplicity of notations, we denote the sets of vertices of X and Y by the same letters X resp. Y.

Definition. A *mapping* $f : X \to Y$ between the sets of vertices of X and Y called a *digraph map (or morphism)* if

$$a \to b \text{ on } X \ \Rightarrow \ f(a) \to f(b) \text{ or } f(a) = f(b) \text{ on } Y.$$

In other words, any arrow of X under the mapping f either goes to an arrow of Y or collapses to a vertex of Y.

Any digraph morphism $f : X \to Y$ induces a mapping $f_* : \Lambda_n(X) \to \Lambda_n(Y)$ as follows: first set

$$f_*(e_{i_0...i_n}) = e_{f(i_0)...f(i_n)},$$

and then extend f_* by linearity to all of $\Lambda_n(X)$.

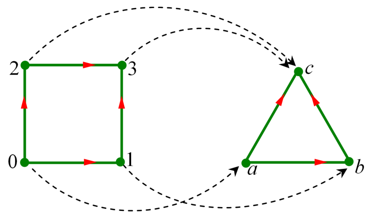
Proposition 1.6. *Let $f : X \to Y$ be a digraph morphism. Then the induced mapping $f_* : \Lambda_n(X) \to \Lambda_n(Y)$ extends to a chain mapping $f_* : \Omega_n(X) \to \Omega_n(Y)$ and, hence, to homomorphism $f_* : H_n(X) \to H_n(Y)$.*

Proof. If $e_{i_0...i_n}$ is irregular then $f_*(e_{i_0...i_n})$ is also irregular. Therefore, f_* maps the space $\mathcal{I}_n(X)$ of irregular paths on X into $\mathcal{I}_n(Y)$. It follows that f_* maps $\mathcal{R}_n(X) = \Lambda_n(X)/\mathcal{I}_n(X)$ into $\mathcal{R}_n(Y)$.

Next, f_* maps the space $\mathcal{A}_n(X)$ of allowed paths into $\mathcal{A}_n(Y)$: if $e_{i_0...i_n}$ is allowed then $i_k \to i_{k+1}$ for all k, which implies that either $f(i_k) \to f(i_{k+1})$ for all k and, hence,

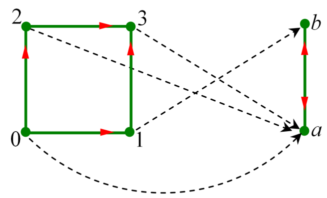
$f_*\left(e_{i_0\ldots i_n}\right)$ is also allowed, or $f\left(i_k\right)=f\left(i_{k+1}\right)$ for some k so that $f_*\left(e_{i_0\ldots i_n}\right)$ is irregular, thus $f_*\left(e_{i_0\ldots i_n}\right)=0$.
Clearly, f_* commutes with ∂, which implies that f_* maps $\Omega_n\left(X\right)$ into $\Omega_n\left(Y\right)$ and f_* is a chain mapping. Consequently, we obtain a homomorphism of homology groups $f_*: H_n(X)\to H_n(Y)$. ■

Example. A triangle e_{abc} and a double arrow e_{aba} are images of a square $e_{013}-e_{023}$ under digraph maps as shown on these pictures:



a digraph map from a square onto a triangle

$$e_{013}-e_{023}\mapsto e_{abc}-e_{acc}=e_{abc}$$



a digraph map from a square onto a double arrow

$$e_{013}-e_{023}\mapsto e_{aba}-e_{aaa}=e_{aba}$$

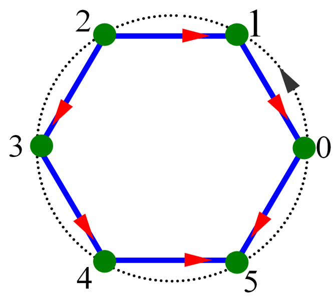
Hence, we can rephrase Theorem 1.5 as follows: Ω_2 is spanned by squares and their morphism images. Or: squares are *basic shapes* of Ω_2.

1.7 Polygons

Let P be a polygon of n vertices $\{0,\ldots,n-1\}$ embedded on a unit circle so that the vertex k has a position $e^{2\pi i\frac{k}{n}}$.

On each edge of the polygon we choose a direction (an arrow) arbitrarily so that P becomes a digraph. For any arrow ξ set

$$\sigma^\xi=\begin{cases}1, & \text{if } \xi \text{ goes counterclockwise,}\\ -1, & \text{if } \xi \text{ goes clockwise.}\end{cases}$$



Consider the following allowed 1-path

$$\sigma=\sum_{\xi\in E}\sigma^\xi e_\xi$$

that is called a *polygonal path*. For example, for the hexagon on the picture we have

$$\sigma=-e_{10}-e_{21}+e_{23}+e_{34}+e_{45}-e_{05}.$$

Lemma 1.7. *We have $\partial\sigma=0$; in particular, σ is ∂-invariant. Moreover,* $\ker\partial|_{\Omega_1}=\langle\sigma\rangle$.

Proof. Any allowed 1-path v can be written in the form

$$v=\sum_{\xi\in E}v^\xi e_\xi,$$

where the sum is take over all arrows ξ. Then $\partial v = \sum_{k\in V} c^k e_k$ so that $\partial v = 0$ if and only if all the coefficients c^k vanish.

Fix a vertex k, let ξ be an arrow between $k-1$ and k, and η be an arrow between k and $k+1$. There are four different possibilities for the directions of these arrows:

$$\overset{k-1}{\bullet} \xrightarrow[\xi]{} \overset{k}{\bullet} \xrightarrow[\eta]{} \overset{k+1}{\bullet} \qquad \overset{k-1}{\bullet} \xleftarrow[\xi]{} \overset{k}{\bullet} \xleftarrow[\eta]{} \overset{k+1}{\bullet} \qquad \overset{k-1}{\bullet} \xrightarrow[\xi]{} \overset{k}{\bullet} \xleftarrow[\eta]{} \overset{k+1}{\bullet} \qquad \overset{k-1}{\bullet} \xleftarrow[\xi]{} \overset{k}{\bullet} \xrightarrow[\eta]{} \overset{k+1}{\bullet}$$

In the first case we have

$$\partial(v^\zeta e_\zeta + v^\eta e_\eta) = v^\zeta(e_k - e_{k-1}) + v^\eta(e_{k+1} - e_k),$$

whence $c^k = v^\xi - v^\eta$. Hence, $c_k = 0$ is equivalent to $v^\zeta = v^\eta$. The second case is similar. In the third case we have

$$\partial(v^\zeta e_\zeta + v^\eta e_\eta) = v^\zeta(e_k - e_{k-1}) + v^\eta(e_k - e_{k+1}),$$

whence $c^k = v^\xi + v^\eta$. Hence, $c^k = 0$ is equivalent to $v^\xi = -v^\eta$. The fours case is similar.

Hence, we obtain the following: $\partial v = 0$ if and only if, for any two successive arrows ξ and η, we have $v^\xi = v^\eta$ if ξ and η have the same directions and $v^\xi = -v^\eta$ if ξ and η have the opposite directions. Clearly, σ satisfies this property, and any other path v with this property is proportional to σ because starting from one arrow, we successively recover the coefficients of v at all other arrows. ∎

Proposition 1.8. *If a polygon P is a triangle or a square then*

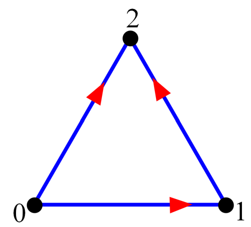
$$|\Omega_2| = 1,\ \Omega_p = \{0\} \text{ for all } p \geq 3 \text{ and } H_p = \{0\} \text{ for all } p \geq 1.$$

Otherwise , if a polygon P is neither a triangle nor a square then

$$\Omega_p = \{0\} \text{ and } H_p = \{0\} \text{ for all } p \geq 2, \text{ while } |H_1| = 1.$$

In short: if P is either triangle or square then a nontrivial space is Ω_2, while otherwise a non-trivial space is H_1. As it follows from Lemma 1.7, in the latter case the generator of H_1 is σ. From homological point of view such polygons represent an 1-dimensional cavity.

Proof. We use Theorem 1.5 in order to compute Ω_2. Let first P be a digraph triangle.



We have $\Omega_1 = \langle e_{01}, e_{02}, e_{12}\rangle,\ \Omega_2 = \langle e_{012}\rangle,$

while $\Omega_p = \mathcal{A}_p = \{0\}$ for $p \geq 3$.

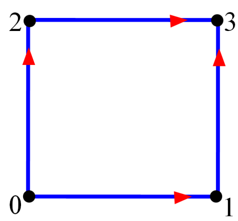
Since $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12}\rangle$ and

$$e_{01} - e_{02} + e_{12} = \partial e_{012},$$

it follows that $H_1 = \{0\}$.

Since $\ker \partial|_{\Omega_2} = 0$, it follows that $H_2 = \{0\}$. Clearly, $H_p = \{0\}$ also for all $p \geq 3$.

Let P be a digraph square:



We have $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}\rangle, \Omega_2 = \langle e_{013} - e_{023}\rangle$

whence $\Omega_p = \mathcal{A}_p = \{0\}$ for $p \geq 3$.

Since $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{13} - e_{23}\rangle$ and

$$e_{01} - e_{02} + e_{13} - e_{23} = \partial\left(e_{013} - e_{023}\right),$$

it follows that $H_1 = \{0\}$.

Since $\ker \partial|_{\Omega_2} = 0$, it follows that $H_2 = \{0\}$. Clearly, $H_p = \{0\}$ for all $p \geq 3$.

Let P be neither triangle nor square. Then P contains no triangles or squares, and we conclude that $\Omega_2 = \{0\}$. Hence, also $\Omega_p = \{0\}$ for all $p \geq 3$, and $|H_p| = 0$ for all $p \geq 2$. For the Euler characteristic, we have

$$\chi = |\Omega_0| - |\Omega_1| = n - n = 0.$$

Since also

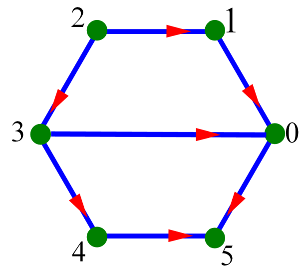
$$\chi = |H_0| - |H_1|$$

and $|H_0| = 1$, we obtain $|H_1| = 1$. ■

Example. By Proposition 1.8, for the above hexagon we have $|\Omega_2| = 0$ and $|H_1| = 1$.

Let us add to the above hexagon a diagonal $3 \to 0$. Then, for the new digraph G, we obtain $|\Omega_2| = 2$ because it has two linearly independent squares

$$u = e_{345} - e_{305} \text{ and } v = e_{230} - e_{210}.$$



It is easy to verify that $\partial(u + v) = \sigma$ so that now σ determines a trivial homology class. Indeed, one can verify that in this case $H_p = \{0\}$ for all $p \geq 1$. One can say that the hexagonal cavity is now filled by two squares.

1.8 Further examples of spaces Ω_p and H_p

Consider an octahedron based on a diamond:

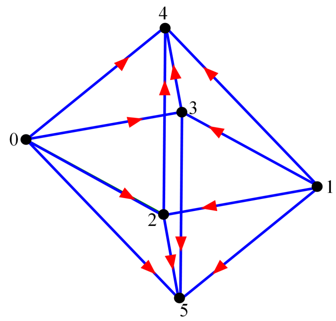
Space Ω_2 is spanned by 8 triangles:

$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle$,

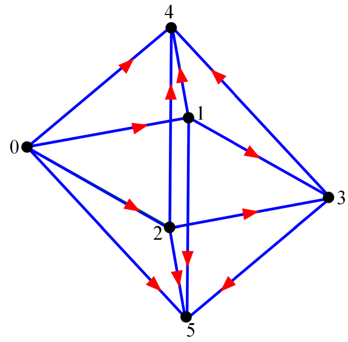
$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \rangle$

$\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for $p = 1$ and $p \geq 3$.

Hence, the octahedron based on a diamond represents a 2-dimensional cavity.



Consider an octahedron based on a square:



$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$

$\Omega_3 = \langle e_{0234} - e_{0134},\ e_{0235} - e_{0135} \rangle,\ \Omega_p = \{0\}\ \forall p \geq 4$

We have $\ker \partial|_{\Omega_2} = \langle u, v \rangle$ where

$$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$$

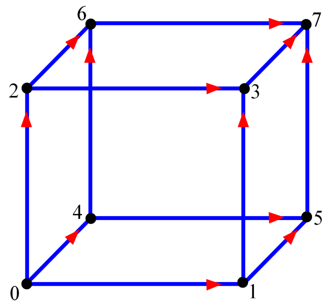
$$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$$

but $H_2 = \{0\}$ because

$$u = \partial\left(e_{0234} - e_{0134}\right) \text{ and } v = \partial\left(e_{0235} - e_{0135}\right).$$

In fact, $H_p = \{0\}$ for all $p \geq 1$.

Consider a 3-cube:



Space Ω_2 is spanned by 6 squares:

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},$$
$$e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467} \rangle$$

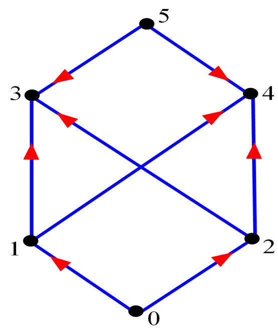
Space Ω_3 is spanned by one 3-cube:

$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \rangle$

$\Omega_p = \{0\}$ for all $p \geq 4$ and $H_p = \{0\}$ for all $p \geq 1$.

1.9 Example of computation of Ω_p and H_p

Consider a hexagon with two diagonals:



We have

$\Omega_0 = \mathcal{A}_0 = \langle e_0, e_1, e_2, e_3, e_4, e_5 \rangle$

$\Omega_1 = \mathcal{A}_1 = \langle e_{01}, e_{02}, e_{13}, e_{14}, e_{23}, e_{24}, e_{53}, e_{54} \rangle$

$\mathcal{A}_2 = \langle e_{013}, e_{014}, e_{023}, e_{024} \rangle$

$\mathcal{A}_p = \{0\}$ for $p \geq 3$.

Hence, $\Omega_p = \{0\}$ and $H_p = \{0\}$ for $p \geq 3$. There are two semi-arrows here: e_{03} and e_{04}. By Theorem 1.5 we obtain

$$|\Omega_2| = |\mathcal{A}_2| - 2 = 2.$$

One sees two squares in this digraph: $e_{013} - e_{023}$ and $e_{014} - e_{024}$ and no triangles. Since these squares are linearly independent, we conclude that

$$\Omega_2 = \langle e_{013} - e_{023}, e_{014} - e_{024} \rangle.$$

Alternatively, one can determine Ω_2 directly by definition. For that, consider the operator $\partial : \mathcal{A}_2 \to \mathcal{R}_1$ and observe that

$$\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\} = \{v \in \mathcal{A}_2 : \partial v = 0 \bmod \mathcal{A}_1\} = \ker\left(\partial : \mathcal{A}_2 \to \mathcal{R}_1 \bmod \mathcal{A}_1\right).$$

Hence, let us first compute $\partial : \mathcal{A}_2 \to \mathcal{R}_1 \bmod \mathcal{A}_1$:

$$\partial e_{013} = e_{13} - e_{03} + e_{01} = -e_{03} \mod \mathcal{A}_1$$

$$\begin{aligned}
\partial e_{014} &= e_{14} - e_{04} + e_{01} = -e_{04} \mod \mathcal{A}_1 \\
\partial e_{023} &= e_{23} - e_{03} + e_{02} = -e_{03} \mod \mathcal{A}_1 \\
\partial e_{024} &= e_{24} - e_{04} + e_{02} = -e_{04} \mod \mathcal{A}_1
\end{aligned}$$

It follows that

$$M := \text{matrix of } \partial : \mathcal{A}_2 \to \mathcal{R}_1 \bmod \mathcal{A}_1 = \begin{array}{cc} \begin{array}{cccc} e_{013} & e_{014} & e_{023} & e_{024} \end{array} & \\ \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} & \begin{array}{c} e_{03} \\ e_{04} \end{array} \end{array}$$

and, hence,

$$\Omega_2 = \ker M = \langle e_{013} - e_{023}, e_{014} - e_{024} \rangle.$$

Let us compute H_1. We have for the basis in Ω_1:

$$\begin{aligned}
\partial e_{01} &= -e_0 + e_1, \quad \partial e_{02} = -e_0 + e_2 \\
\partial e_{13} &= -e_1 + e_3, \quad \partial e_{14} = -e_1 + e_4 \\
\partial e_{23} &= -e_2 + e_3, \quad \partial e_{23} = -e_2 + e_3 \\
\partial e_{53} &= -e_5 + e_3, \quad \partial e_{54} = -e_5 + e_4
\end{aligned}$$

Therefore,

$$M := \text{matrix of } \partial|_{\Omega_1} = \begin{array}{cc} \begin{array}{cccccccc} e_{01} & e_{02} & e_{13} & e_{14} & e_{23} & e_{24} & e_{53} & e_{54} \end{array} & \\ \begin{pmatrix} -1 & -1 & & & & & & \\ 1 & & -1 & -1 & & & & \\ & 1 & & & -1 & -1 & & \\ & & 1 & & 1 & & 1 & \\ & & & 1 & & 1 & & 1 \\ & & & & & & -1 & -1 \end{pmatrix} & \begin{array}{c} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \end{array}.$$

Computation shows that $\operatorname{rank} M = 5$ and, hence, $\dim \ker M = 3$; moreover,

$$\ker \partial|_{\Omega_1} = \ker M = \langle e_{01} - e_{02} + e_{13} - e_{23},\ e_{01} - e_{02} + e_{14} - e_{24},\ e_{13} - e_{14} - e_{53} + e_{54} \rangle.$$

Since

$$\begin{aligned}
\operatorname{Im} \partial|_{\Omega_2} &= \langle \partial\left(e_{013} - e_{023}\right), \partial\left(e_{014} - e_{024}\right) \rangle \\
&= \langle e_{13} - e_{01} - e_{23} + e_{02},\ e_{14} - e_{01} - e_{24} - e_{02} \rangle
\end{aligned}$$

it follows that

$$H_1 = \ker \partial|_{\Omega_1} / \operatorname{Im} \partial|_{\Omega_2} = \langle e_{13} - e_{14} - e_{53} + e_{54} \rangle.$$

Since

$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = 6 - 8 + 2 = 0$$

and

$$\chi = |H_0| - |H_1| + |H_2| = 1 - 1 + |H_2| = |H_2|,$$

it follows that also $|H_2| = 0$.

Problem. Devise an efficient algorithm/software for computation of the spaces Ω_p for arbitrary digraphs. Is there a way of computing directly $\dim \Omega_p$ without computing the bases of Ω_p?

2 Join of digraphs and Künneth formula

2.1 Augmented chain complex

In this section we use the augmented chain complex

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \Lambda_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \dots \tag{2.1}$$

with $\Lambda_{-1} = \mathbb{K}$ and $\Lambda_{-2} = \{0\}$. The operator $\partial : \Lambda_0 \to \Lambda_{-1}$ is now define by

$$\partial e_i = e = \text{the unity of } \mathbb{K},$$

which matches the definition

$$\partial e_{i_0 \dots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \dots \widehat{i_q} \dots i_p}$$

also for $p = 0$. As in the proof of Lemma 1.1, we have $\partial^2 = 0$ also for (2.1).
Based on (2.1) we obtain also the regularized chain complex

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \mathcal{R}_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \dots \tag{2.2}$$

and the *augmented path chain complex* of a digraph G:

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots , \tag{2.3}$$

where $\mathcal{R}_{-1} = \Omega_{-1} = \mathbb{K}$. The homology groups of (2.3) are called the *reduced* homology groups of G and are denoted by $\widetilde{H}_p$. Clearly, we have

$$\widetilde{H}_{-1} = \{0\} \ \text{ and } \ \widetilde{H}_p = H_p \text{ for } p \geq 1.$$

Since

$$\ker \partial|_{\Omega_0} = \langle e_i - e_0 \rangle_{i \geq 1} \cong \Omega_0 / \mathbb{K},$$

it follows that

$$\widetilde{H}_0 = H_0 / \mathbb{K}.$$

Define the reduced Betti numbers: $\widetilde{\beta}_p = |\widetilde{H}_p|$. We have

$$\widetilde{\beta}_{-1} = 0, \quad \widetilde{\beta}_p = \beta_p \text{ for } p \geq 1, \ \text{ and } \ \widetilde{\beta}_0 = \beta_0 - 1.$$

For a disjoint union $X \sqcup Y$ of two digraphs we have by (1.4)

$$\widetilde{\beta}_p (X \sqcup Y) = \widetilde{\beta}_p (X) + \widetilde{\beta}_p (Y) + \mathbf{1}_{\{p=0\}}. \tag{2.4}$$

2.2 Join of paths

Let X and Y be two disjoint finite sets.

Definition. Let $p, q \geq -1$. For any paths $u \in \Lambda_p(X)$ and $v \in \Lambda_q(Y)$, define their *join* $u * v$ as a $(p+q+1)$-path on $X \sqcup Y$ as follows: first define it for elementary paths by

$$e_{i_0 \dots i_p} * e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q},$$

where $i_0, \dots, i_p \in X$ and $j_0, \dots, j_q \in Y$, and then extend this definition by linearity to all u and v.

For example, $e_{i_0 \dots i_p} * e = e_{i_0 \dots i_p}$. Clearly, the join of regular paths is regular.

Lemma 2.1. (Product rule for join) *Let $p, q \geq -1$. For all $u \in \Lambda_p(X)$ and $v \in \Lambda_q(Y)$ we have*

$$\partial (u * v) = (\partial u) * v + (-1)^{p+1} u * \partial v. \tag{2.5}$$

Proof. It suffices to prove (2.5) for $u = e_{i_0 \dots i_p}$ and $v = e_{j_0 \dots j_q}$. We have

$$\begin{aligned}
\partial (u * v) &= \partial e_{i_0 \dots i_p j_0 \dots j_q} \\
&= e_{i_1 \dots i_p j_0 \dots j_q} - e_{i_0 i_2 \dots i_p j_0 \dots j_q} + \dots + (-1)^p e_{i_0 \dots i_{p-1} j_0 \dots j_q} \\
&\quad + (-1)^{p+1} \left(e_{i_0 \dots i_p j_1 \dots j_q} - e_{i_0 \dots i_p j_0 j_2 \dots j_q} + \dots + (-1)^q e_{i_0 \dots i_p j_0 \dots j_{q-1}} \right) \\
&= \left(\partial e_{i_0 \dots i_p} \right) * e_{j_0 \dots j_q} + (-1)^{p+1} e_{i_0 \dots i_p} * \partial e_{j_0 \dots j_q},
\end{aligned}$$

which was to be proved. ■

Note that (2.5) is not true under the convention $\Lambda_{-1} = \{0\}$. Indeed, in this case we have

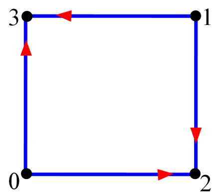
$$\partial (e_i * e_j) = \partial e_{ij} = e_j - e_i$$

while $\partial e_i * e_j = 0$ and $e_i * \partial e_j = 0$ so that the right hand side of (2.5) vanishes.

Let now X, Y be two digraphs.

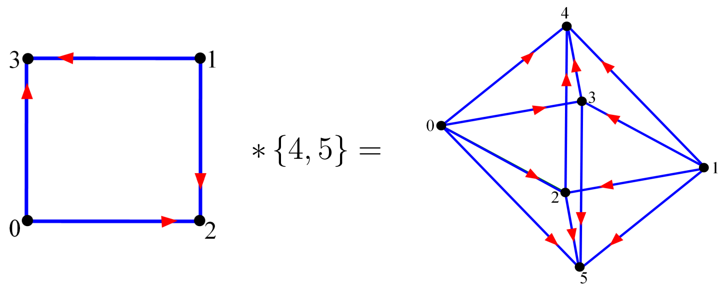
Definition. The *join* $X * Y$ of the digraphs X, Y is a digraph whose set of vertices is a disjoint union of the sets of vertices of X and Y, and the set of arrows consists of all arrows of X and Y as well as from all arrows $x \to y$ where $x \in X$ and $y \in Y$.

Example. For example, if $X = Y = \{\cdot, \cdot\}$ are digraphs with two vertices and no arrows, then $X * Y$ is a *diamond*:

$\{0, 1\} * \{2, 3\} =$ 

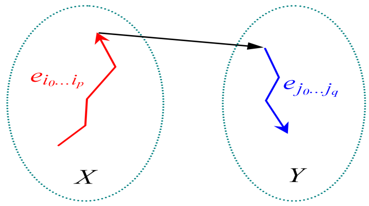
a diamond

If X is the above diamond and $Y = \{\cdot, \cdot\}$ then $X * Y$ is an *octahedron*:



an octahedron based on diamond

Denote $Z = X * Y$. It is easy to see that the joint of two allowed elementary paths is allowed.



The join $e_{i_0 \ldots i_p} * e_{j_0 \ldots j_q}$ of allowed paths $e_{i_0 \ldots i_p}$ and $e_{j_0 \ldots j_q}$

Consequently, we have

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \Rightarrow u * v \in \mathcal{A}_{p+q+1}(Z). \tag{2.6}$$

Lemma 2.2. *Let $r = p + q + 1$, where $p, q \geq -1$.*
(a) *We have*

$$u \in \Omega_p(X) \text{ and } v \in \Omega_q(Y) \Rightarrow u * v \in \Omega_r(Z).$$

(b) *The join $u * v$ is well defined for the reduced homology classes:*

$$u \in \widetilde{H}_p(X) \text{ and } v \in \widetilde{H}_q(Y) \Rightarrow u * v \in \widetilde{H}_r(Z).$$

Proof. (a) If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u * v$ is allowed by (2.6). Since ∂u and ∂v are allowed, we obtain by (2.5) that $\partial (u * v)$ is also allowed, which proves that $u * v \in \Omega_r(Z)$.
(b) Recall that a homology class of $\widetilde{H}_n$ consists of closed n-paths modulo boundaries; that is, two closed n-paths w_1 and w_2 determine the same homology class if $w_1 = w_2 + \partial\omega$ for some $\omega \in \Omega_{n+1}$; in this case we write $w_1 \sim w_2$.
If $u \in \widetilde{H}_p(X)$ and $v \in \widetilde{H}_q(Y)$ then they have representatives $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ that are closed paths, that is, $\partial u = 0$, $\partial v = 0$. Then by (2.5) we obtain that also $\partial(u * v) = 0$ so that $u * v$ determines a homology class in $\widetilde{H}_r(Z)$.
Let us show that $u * v$ as a homology class does not depend on the choice of representatives. Let $u' \sim u$ that is, $u' = u + \partial\omega$ for some $\omega \in \Omega_{p+1}(X)$. Then by (2.5)

$$u' * v = u * v + \partial\omega * v = u * v + \partial(\omega * v) - (-1)^{p+2}\omega * \partial v = u * v + \partial(\omega * v)$$

so that $u' * v \sim u * v$. In the same way, the homology class of $u * v$ does not change if we replace v by $v' \sim v$. Therefore, the operation $*$ is well defined in homologies. ∎

2.3 Künneth formula for join

Theorem 2.3. (Künneth formula for join) *Let X, Y be two digraphs and $Z = X * Y$. Then, for any $r \geq -1$, there is an isomorphism*

$$\Omega_r(Z) \cong \bigoplus_{\{p,q\geq -1: p+q+1=r\}} \Omega_p(X) \otimes \Omega_q(Y), \tag{2.7}$$

hat is given by the map $u \otimes v \mapsto u * v$ for $u \in \Omega_p(X)$ and $v \in \Omega_p(Y)$. Besides, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q+1=r\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y) \tag{2.8}$$

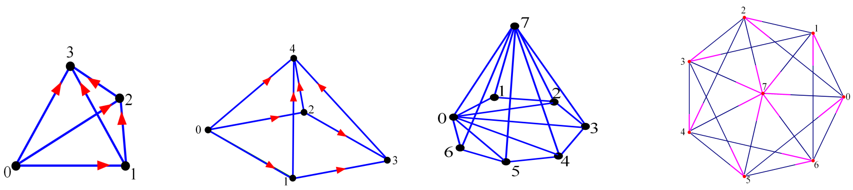
and

$$\widetilde{\beta}_r(Z) = \sum_{\{p,q\geq 0: p+q+1=r\}} \widetilde{\beta}_p(X)\,\widetilde{\beta}_q(Y). \tag{2.9}$$

The identity (2.7) essentially means that any path in $\Omega_r(Z)$ can be obtained as linear combination of joins $u * v$ where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with $p + q + 1 = r$, and (2.8) means the same for homology classes. The restrictions $p, q \geq 0$ in (2.8) and (2.9) as opposed to $p, q \geq -1$ in (2.7) come from the fact that $\widetilde{H}_{-1} = \{0\}$.

Example. Let Y consist of a single vertex. In this case the join $X * Y$ is called a *cone* over X. Since all the reduced homology groups $\widetilde{H}_q(Y)$ are trivial, the cone $X * Y$ is also homologically trivial by (2.8).

For example, the following digraphs are cones and, hence, they are homologically trivial.



Example. Let Y consist of two disjoint vertices: $Y = \{\cdot, \cdot\}$. Then the join $X * Y$ is called a *suspension* of X and is denoted by $\operatorname{sus} X$. Since $\widetilde{\beta}_0(Y) = 1$ and $\widetilde{\beta}_q(Y) = 0$ for $q \geq 1$, we obtain from (2.9) that

$$\widetilde{\beta}_r(\operatorname{sus} X) = \widetilde{\beta}_{r-1}(X). \tag{2.10}$$

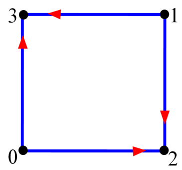
Since $|\Omega_{-1}(Y)| = 1$ and $|\Omega_0(Y)| = 2$, it follows from (2.7) that

$$\begin{aligned} |\Omega_r(\operatorname{sus} X)| &= \sum_{\{p,q\geq -1: p+q+1=r\}} |\Omega_p(X)|\,|\Omega_q(Y)| \\ &= |\Omega_r(X)| + 2\,|\Omega_{r-1}(X)|. \end{aligned} \tag{2.11}$$

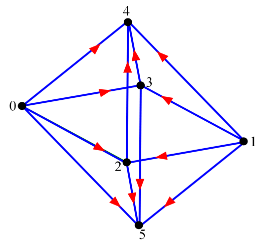
Consider a family $\{S^n\}_{n=0}^{\infty}$ of digraphs that is defined inductively as follows: $S^0 = \{\cdot, \cdot\}$ and

$$S^{n+1} = \operatorname{sus} S^n.$$

We refer to S^n as a digraph *n-sphere*. For example, S^1 is a diamond and S^2 is an octahedron as in the example above:



a diamond S^1



an octahedron S^2

Since by (2.10)

$$\widetilde{\beta}_r\left(S^n\right) = \widetilde{\beta}_{r-1}\left(S^{n-1}\right),$$

it follows by induction that the only non-trivial Betti number of S^n is

$$\widetilde{\beta}_n(S^n) = 1.$$

It follows from (2.11) that

$$|\Omega_p(S^n)| = 0 \text{ for } p > n \text{ and } |\Omega_n(S^n)| = 2^{n+1}.$$

Using (2.8) we obtain

$$\widetilde{H}_1(S^1) = \widetilde{H}_0\left(\{0,1\}\right) * \widetilde{H}_0\left(\{2,3\}\right) = \langle (e_1 - e_0) * (e_3 - e_2) \rangle = \langle e_{02} + e_{13} - e_{03} - e_{12} \rangle,$$

that is, $H_1(S^1)$ is generated by the polygonal path of the diamond, which matches Lemma 1.7 and Proposition 1.8.

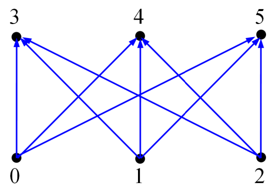
Similarly we have

$$\begin{aligned}\widetilde{H}_2(S^2) &= \widetilde{H}_1\left(S^1\right) * \widetilde{H}_0\left(\{4,5\}\right)\\ &= \langle (e_{02} + e_{13} - e_{03} - e_{12}) * (e_4 - e_5) \rangle\\ &= \langle e_{024} + e_{134} + e_{035} + e_{125} - e_{034} - e_{124} - e_{025} - e_{135} \rangle .\end{aligned}$$

Using (2.7) we obtain

$$\begin{aligned}\Omega_2(S^2) &= \Omega_1(S^1) * \Omega_0\left(\{4,5\}\right)\\ &= \langle e_{02}, e_{03}, e_{12}, e_{13} \rangle * \{e_4, e_5\} = \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135} \rangle .\end{aligned}$$

Example. Let $X = Y = \{\cdot, \cdot, \cdot\}$. Consider the digraph $G = X * Y$ that can be regarded as a directed analogue of the bipartite graph $K_{3,3}$:



The only non-trivial Betti number of G is

$$\widetilde{\beta}_1(G) = \widetilde{\beta}_0(X)\widetilde{\beta}_0(Y) = 4.$$

The generators of $\widetilde{H}_1(G)$ are

$$\begin{aligned}\widetilde{H}_1(G) &= \widetilde{H}_0\left(\{0,1,2\}\right) * \widetilde{H}_0\left(\{3,4,5\}\right) = \langle e_2 - e_0, e_1 - e_0 \rangle * \langle e_4 - e_3, e_5 - e_3 \rangle\\ &= \langle e_{24} + e_{03} - e_{23} - e_{04},\ e_{25} + e_{03} - e_{23} - e_{05},\\ &\qquad e_{14} + e_{03} - e_{13} - e_{04},\ e_{15} + e_{03} - e_{13} - e_{05} \rangle.\end{aligned}$$

2.4 Preparation for the proof of the Künneth formula

Denote by $\mathcal{R}_*(V)$ the linear space of all finite linear combinations with coefficients from $\mathbb{K}$ of regular elementary paths on V of any length $p \geq -1$. In particular, $\mathcal{R}_*$ contains all $\mathcal{R}_p$. Define a $\mathbb{K}$-valued bilinear form $\langle u, v \rangle$ for all $u, v \in \mathcal{R}_*$ as follows: for elementary regular paths $e_{i_0 \dots i_p}$ and $e_{j_0 \dots j_q}$ set

$$\left\langle e_{i_0 \dots i_p}, e_{j_0 \dots j_q} \right\rangle = \delta^{i_0 \dots i_p}_{j_0 \dots j_q} := \begin{cases} 1_{\mathbb{K}}, & i_0 \dots i_p = j_0 \dots j_q \\ 0_{\mathbb{K}}, & \text{otherwise.} \end{cases} \tag{2.12}$$

In particular, $\langle u, v \rangle = 0$ if $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$ with $p \neq q$. If $\mathbb{K} = \mathbb{R}$ then $\langle \cdot, \cdot \rangle$ is an inner product on each space $\mathcal{R}_p$.

Let X, Y be two disjoint finite sets.

Lemma 2.4. *For all $u, \varphi \in \mathcal{R}_*(X)$, $v, \psi \in \mathcal{R}_*(Y)$, we have*

$$\langle u * v, \varphi * \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle. \tag{2.13}$$

Proof. Due to bilinearity it suffices to prove (2.13) if u, v, φ, ψ are elementary paths, say

$$u = e_x, \quad \varphi = e_{x'}, \qquad v = e_y, \quad \psi = e_{y'}$$

where $x = i_0 \dots i_p$, $x' = i'_0 \dots i'_{p'}$, $y = j_0 \dots j_q$, $y' = j'_0 \dots j'_{q'}$. Using (2.12) we obtain

$$\begin{aligned} \langle u * v, \varphi * \psi \rangle &= \langle e_x * e_y, e_{x'} * e_{y'} \rangle = \langle e_{xy}, e_{x'y'} \rangle \\ &= \delta^{x'y'}_{xy} = \delta^{x'}_{x} \delta^{y'}_{y} = \langle e_x, e_{x'} \rangle \langle e_y, e_{y'} \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle. \end{aligned}$$

∎

For any digraph G, we denote by $A(G)$ the set of all allowed elementary paths on G, and by $R(G)$ – the set of all regular elementary paths on G.

In the rest of this section, X, Y are two digraphs, and $Z = X * Y$.

Lemma 2.5. *Any path $w \in \Omega_*(Z)$ admits a representation*

$$w = \sum_{x \in A(X)} e_x * a^x = \sum_{y \in A(Y)} b^y * e_y, \tag{2.14}$$

where $a^x \in \Omega_*(Y)$ and $b^y \in \Omega_*(X)$ are uniquely determined.

Proof. Since any allowed elementary path on Z is a join of elementary paths on X and Y, we see that any $w \in \mathcal{A}_*(Z)$ admits a representation

$$w = \sum_{x \in A(X),\ y \in A(Y)} c^{xy} e_x * e_y, \tag{2.15}$$

where the coefficients $c^{xy} \in \mathbb{K}$ are uniquely determined. It follows from (2.15) that

$$w = \sum_{x \in A(X)} e_x * a^x, \tag{2.16}$$

where

$$a^x = \sum_{y\in A(Y)} c^{xy} e_y \in \mathcal{A}_*(Y).$$

Clearly, a^x are uniquely determined.

Assume now that $w \in \Omega_*(Z)$ and show that $a^x \in \Omega_*(Y)$. Let us define the coefficients $\varepsilon_{x'}^x \in \{0,1,-1\}$ by

$$\partial e_x = \sum_{x'\in R(X)} \varepsilon_x^{x'} e_{x'}. \tag{2.17}$$

Also, if $x \in A_p(X)$ then set $\sigma_x = (-1)^{p+1}$ so that by the product rule (2.5)

$$\begin{aligned}\partial(e_x * a^x) &= (\partial e_x) * a^x + \sigma_x e_x * (\partial a^x) \\ &= \sum_{x'\in R(X)} \varepsilon_x^{x'} e_{x'} * a^x + \sigma_x e_x * (\partial a^x).\end{aligned}$$

Substituting into (2.16) we obtain

$$\partial w = \sum_{x\in A(X)} \sum_{x'\in R(X)} \varepsilon_x^{x'} e_{x'} * a^x + \sum_{x\in A(X)} \sigma_x e_x * \partial a^x.$$

Switching in the double sum the notations x and x' and interchanging the summation signs, we obtain

$$\begin{aligned}\partial w &= \sum_{x\in R(X)} \sum_{x'\in A(X)} \varepsilon_{x'}^x e_x * a^{x'} + \sum_{x\in A(X)} \sigma_x e_x * \partial a^x \\ &= \sum_{x\in A(X)} \sum_{x'\in A(X)} \varepsilon_{x'}^x e_x * a^{x'} + \sum_{x\in R(X)\setminus A(X)} \sum_{x'\in A(X)} \varepsilon_{x'}^x e_x * a^{x'} + \sum_{x\in A(X)} \sigma_x e_x * \partial a^x \\ &= \sum_{x\in A(X)} e_x * \left(\sum_{x'\in A(X)} \varepsilon_{x'}^x a^{x'} + \sigma_x \partial a^x \right) \qquad (2.18) \\ &+ \sum_{x\in R(X)\setminus A(X)} e_x * \left(\sum_{x'\in A(X)} \varepsilon_{x'}^x a^{x'} \right). \qquad (2.19)\end{aligned}$$

Note that any elementary path of the full expansion of the sum (2.19) has a non-allowed X-part, while that of (2.18) has the allowed X-part. Therefore, there is no cross cancellation between the elementary paths of (2.18) and (2.19). Since their sum ∂w is allowed, it follows that the sum (2.19), consisting only of non-allowed paths, must vanish.

On the other hand, since $\partial w \in \Omega_*(Z)$, we have analogously to (2.16) a representation

$$\partial w = \sum_{x\in A(X)} e_x * \widetilde{a}^x,$$

where $\widetilde{a}^x \in \mathcal{A}_*(Y)$. Comparison with (2.18) yields

$$\widetilde{a}^x = \sum_{x'\in A(X)} \varepsilon_{x'}^x a^{x'} + \sigma_x \partial a^x.$$

Since $a^{x'} \in \mathcal{A}_* (Y)$, it follows that $\partial a^x \in \mathcal{A}_* (Y)$, which proves that $a^x \in \Omega_* (Y)$.
The second identity in (2.14) is proved similarly. ■
For any digraph G, set

$$\Omega_p^{\perp} (G) = \{u \in \mathcal{A}_p (G) : \langle u, v \rangle = 0 \text{ for all } v \in \Omega_p (G)\} . \tag{2.20}$$

Lemma 2.6. *Let $r = p + q + 1$. Then*

$$u \in \Omega_p^{\perp} (X) \ \ \textit{and} \ \ v \in \mathcal{A}_q (Y) \ \Rightarrow \ u * v \in \Omega_r^{\perp} (Z)$$

and

$$u \in \mathcal{A}_p (X) \ \ \textit{and} \ \ v \in \Omega_q^{\perp} (Y) \ \Rightarrow \ u * v \in \Omega_r^{\perp} (Z) .$$

Proof. Let us prove the first claim that can be restated as follows: if $u \in \Omega_p^{\perp} (X)$ and $v \in \mathcal{A}_q (Y)$ then

$$\langle u * v, w \rangle = 0 \text{ for any } w \in \Omega_r (Z) .$$

By Lemma 2.5, w is a sum of the joins $\varphi * \psi$ where $\varphi \in \Omega_* (X)$ and $\psi \in \mathcal{A}_* (Y)$. Hence, it suffices to prove that

$$\langle u * v, \varphi * \psi \rangle = 0, \tag{2.21}$$

assuming that $\varphi \in \Omega_{p'} (X)$ and $\psi \in \mathcal{A}_{q'} (Y)$. By (2.13) we have

$$\langle u * v, \varphi * \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle . \tag{2.22}$$

If $p = p'$ then $\langle u, \varphi \rangle = 0$ by the hypothesis $u \in \Omega_p^{\perp} (X)$. If $p \neq p'$ then $\langle u, \varphi \rangle = 0$ holds trivially. Hence, in the both cases the right hand side of (2.22) vanishes, which proves (2.21).
■

Now we can state and prove the main technical result of this section.

Theorem 2.7. *Let X, Y be two digraphs and $Z = X * Y$. Any path $w \in \Omega_r (Z)$ with $r \geq -1$ admits a representation in the form*

$$w = \sum_{j=1}^{k} u_j * v_j \tag{2.23}$$

for some finite k, with some $u_j \in \Omega_{p_j} (X)$ and $v_j \in \Omega_{q_j} (Y)$, where $p_j, q_j \geq -1$ and $p_j + q_j + 1 = r$.

Proof. Given two subspaces $U \subset \mathcal{A}_p (X)$ and $V \subset \mathcal{A}_q (Y)$, denote by $U * V$ the subspace of $\mathcal{A}_r (Z)$ that is spanned by all joins $u * v$ with $u \in U$ and $v \in V$.
In the proof we use the following properties of join.

(i) If $u \in \mathcal{A}_p (X)$ and $v \in \mathcal{A}_q (Y)$ then $u * v \in \mathcal{A}_r (Z)$ where $r = p + q + 1$ (cf. (2.6)).

(ii) Conversely, any allowed elementary path on Z is a join of allowed elementary paths on X and Y, which implies that

$$\mathcal{A}_r (Z) = \sum_{\{p,q \geq -1 : \, p+q+1=r\}} \mathcal{A}_p (X) * \mathcal{A}_q (Y) . \tag{2.24}$$

(iii) If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u * v \in \Omega_r(Z)$ (Lemma 2.2).

(iv) If $u \in \Omega_p^{\perp}(X)$ and $v \in \mathcal{A}_q(Y)$ then $u * v \in \Omega_r^{\perp}(Z)$ (Lemma 2.6).

For any $r \geq -1$ set

$$\widetilde{\Omega}_r(Z) = \sum_{\{p,q \geq -1:\, p+q+1=r\}} \Omega_p(X) * \Omega_q(Y). \tag{2.25}$$

By Lemma 2.2, we have

$$\widetilde{\Omega}_r(Z) \subset \Omega_r(Z).$$

The existence of the representation (2.23) is equivalent to the opposite inclusion, that is, to the identity

$$\widetilde{\Omega}_r(Z) = \Omega_r(Z). \tag{2.26}$$

Consider first the case when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$. We use the fact that a linear space with a $\mathbb{R}$- or $\mathbb{Q}$-scalar product is represented as a direct sum of a subspace with its orthogonal complement. Fix some $p, q \geq -1$ and set $r = p + q + 1$. For any $u \in \mathcal{A}_p(X)$ consider a decomposition

$$u = u_{\Omega} + u_{\perp}$$

where $u_{\Omega} \in \Omega_p(X)$ and $u_{\perp} \in \Omega_p^{\perp}(X)$; similarly, for any $v \in \mathcal{A}_q(Y)$ we have a decomposition

$$v = v_{\Omega} + v_{\perp}$$

with $v_{\Omega} \in \Omega_q(Y)$ and $v_{\perp} \in \Omega_q^{\perp}(Y)$. It follows that

$$u * v = u_{\Omega} * v_{\Omega} + u_{\Omega} * v_{\perp} + u_{\Omega} * v_{\perp} + u_{\perp} * v_{\perp}.$$

Here $u_{\Omega} * v_{\Omega} \in \widetilde{\Omega}_r(Z)$, while by Lemma 2.6 all other terms in the right hand side belong to $\Omega_r^{\perp}(Z)$, whence it follows that

$$u * v \in \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z). \tag{2.27}$$

It follows from (2.27) and (2.24) that

$$\mathcal{A}_r(Z) = \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z).$$

Comparing with the decomposition

$$\mathcal{A}_r(Z) = \Omega_r(Z) \oplus \Omega_r^{\perp}(Z),$$

we obtain (2.26).

Consider now the general case of an arbitrary field $\mathbb{K}$. It suffices to prove that

$$|\Omega_r(Z)| \leq |\widetilde{\Omega}_r(Z)|. \tag{2.28}$$

Let us introduce the following notation:

$$\begin{aligned} a_p &= |\mathcal{A}_p(X)|, \quad a_q = |\mathcal{A}_q(Y)|, \quad a_r = |\mathcal{A}_r(Z)|, \\ \omega_p &= |\Omega_p(X)|, \quad \omega_q = |\Omega_q(Y)|, \quad \omega_r = |\Omega_r(Z)|, \end{aligned}$$

and observe that

$$\left|\Omega_p^{\perp}(X)\right| = a_p - \omega_p, \quad \left|\Omega_q^{\perp}(Y)\right| = a_q - \omega_q, \quad \left|\Omega_r^{\perp}(Z)\right| = a_r - \omega_r. \tag{2.29}$$

Since $\mathcal{A}_p(X) * \mathcal{A}_q(Y)$ has a basis $\{e_x * e_y\}$ where e_x is any elementary allowed p-path on X and e_y is any elementary allowed q-path on Y, we have

$$|\mathcal{A}_p(X) * \mathcal{A}_q(Y)| = a_p a_q.$$

Moreover, $\mathcal{A}_r(Z)$ has a basis $\{e_x * e_y\}$ where e_x and e_y are as above and $p, q \geq -1$ take all values such that $p + q + 1 = r$, which yields

$$a_r = \sum_{\{p.q \geq -1, p+q+1=r\}} a_p a_q. \tag{2.30}$$

In particular, the sum of subspaces in (2.24) is direct.

Before we can proceed further, let us prove the following claim.

Claim. For any two subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$, we have

$$|U * V| = |U|\,|V| \tag{2.31}$$

and

$$(U * \mathcal{A}_q(Y)) \cap (\mathcal{A}_p(X) * V) = U * V. \tag{2.32}$$

Let $u_1, u_2, ..u_k$ be a basis in U and $v_1, ...v_l$ be a basis in V. Then $U * V$ is spanned by all products $u_i * v_j$, so that

$$|U * V| \leq kl.$$

Let us complement the basis $\{u_i\}$ to a basis in $\mathcal{A}_p(X)$ by adding additional paths $u'_1, ..., u'_{k'}$, and, similarly, complement $\{v_j\}$ to a basis in $\mathcal{A}_q(Y)$ by adding $v'_1, ..., v'_{l'}$. Set

$$U' = \langle u'_i \rangle \text{ and } V' = \langle v'_j \rangle.$$

Then

$$\begin{aligned} \mathcal{A}_p(X) * \mathcal{A}_q(Y) &= (U + U') * (V + V') \\ &= U * V + U * V' + U' * V + U' * V', \end{aligned} \tag{2.33}$$

whence

$$\begin{aligned} a_p a_q &= |\mathcal{A}_p(X) * \mathcal{A}_q(Y)| \\ &\leq |U * V| + |U * V'| + |U' * V| + |U' * V'| \\ &\leq kl + kl' + k'l + k'l' = (k + k')(l + l') = a_p a_q. \end{aligned} \tag{2.34}$$

Since the left and right hand sides are equal, we must have the equality case in all inequalities above, whence

$$|U * V| = kl,$$

which proves (2.31).

It also follows from (2.34) that the sum at the right hand side of (2.33) is a direct sum of subspaces, that is,

$$\mathcal{A}_p(X) * \mathcal{A}_q(Y) = (U * V) \oplus (U * V') \oplus (U' * V) \oplus (U' * V'). \tag{2.35}$$

Therefore,

$$U * \mathcal{A}_q(Y) = U * (V + V') = U * V + U * V' = (U * V) \oplus (U * V')$$

and

$$\mathcal{A}_p(X) * V = (U + U') * V = (U * V) \oplus (U' * V).$$

Hence, if $U * \mathcal{A}_q(Y)$ and $\mathcal{A}_p(X) * V$ have a common vector then it has a form $w_1 + u = w_2 + v$ where

$$w_1, w_2 \in U * V, \quad u \in U * V', \quad v \in U' * V$$

whence

$$(w_1 - w_2) + u - v = 0.$$

It follows from (2.35) that $w_1 - w_2 = u = v = 0$ whence (2.32) follows.

By Lemma 2.6, we have

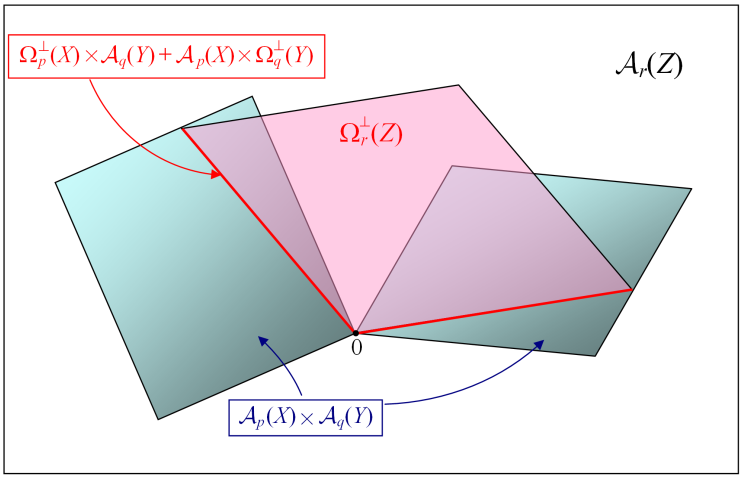
$$\Omega_p^{\perp}(X) * \mathcal{A}_q(Y) \subset \Omega_r^{\perp}(Z)$$

and

$$\mathcal{A}_p(X) * \Omega_q^{\perp}(Y) \subset \Omega_r^{\perp}(Z)$$

so that

$$\sum_{\{p,q \geq -1, p+q+1=r\}} \left[(\Omega_p^{\perp}(X) * \mathcal{A}_q(Y)) + (\mathcal{A}_p(X) * \Omega_q^{\perp}(Y)) \right] \subset \Omega_r^{\perp}(Z) \tag{2.36}$$



Space $\mathcal{A}_r(Z)$ and its subspaces $\Omega_r^{\perp}(Z)$, $\mathcal{A}_p(X) \times \mathcal{A}_q(Y)$ (two instances for different pairs of p, q), and $\Omega_p^{\perp}(X) \times \mathcal{A}_q(Y) + \mathcal{A}_p(X) \times \Omega_q^{\perp}(Y)$.

Note that the space in the square brackets in (2.36) is a subspace of $\mathcal{A}_p(X) * \mathcal{A}_q(Y)$. Since the sum of subspaces in (2.24) is direct, it follows that also the sum of subspaces in (2.36) is direct, which implies the inequality

$$\sum_{\{p,q \geq -1, p+q+1=r\}} \left| (\Omega_p^{\perp}(X) * \mathcal{A}_q(Y)) + (\mathcal{A}_p(X) * \Omega_q^{\perp}(Y)) \right| \leq \left| \Omega_r^{\perp}(Z) \right|. \tag{2.37}$$

By (2.32), the subspaces $\Omega_p^\perp(X) * \mathcal{A}_q(Y)$ and $\mathcal{A}_p(X) * \Omega_q^\perp(Y)$ have intersection $\Omega_p^\perp(X) * \Omega_q^\perp(Y)$, whence

$$\begin{aligned} &\left|(\Omega_p^\perp(X) * \mathcal{A}_q(Y)) + (\mathcal{A}_p(X) * \Omega_q^\perp(Y))\right| \\ &\quad = \left|\Omega_p^\perp(X) * \mathcal{A}_q(Y)\right| + \left|\mathcal{A}_p(X) * \Omega_q^\perp(Y)\right| - \left|\Omega_p^\perp(X) * \Omega_q^\perp(Y)\right|. \end{aligned} \tag{2.38}$$

Using (2.29), we obtain that the right hand side of (2.38) is equal to

$$(a_p - \omega_p)\, a_q + a_p\, (a_q - \omega_q) - (a_p - \omega_p)\,(a_q - \omega_q) = a_p a_q - \omega_p \omega_q.$$

Substituting this into (2.37) yields

$$\sum_{p+q+1=r} (a_p a_q - \omega_p \omega_q) \le a_r - \omega_r,$$

which together with (2.30) implies that

$$\omega_r \le \sum_{p+q+1=r} \omega_p \omega_q.$$

Finally, we are left to observe that, by (2.25),

$$\sum_{p+q+1=r} \omega_p \omega_q = |\widetilde{\Omega}_r(Z)|,$$

which proves (2.28). ■

2.5 Proof of the Künneth formula

Let us first recall some notions from homological algebra. Let $A_* = \{A_p\}_{p\ge 0}$ be a chain complex that consists of linear spaces A_p over $\mathbb{K}$, with the boundary operator ∂_A, and $B_* = \{B_q\}_{q\ge 0}$ be a similar chain complex with the boundary operator ∂_B.
Consider the *tensor product* of the chain complexes

$$C_* = A_* \otimes B_* = \{C_r\}_{r\ge 0}$$

that consists of linear spaces

$$C_r = \bigoplus_{\{p,q\ge 0:\, p+q=r\}} A_p \otimes B_q$$

with the boundary operator ∂_C that is defined by

$$\partial_C\,(u \otimes v) = (\partial_A u) \otimes v + (-1)^p\, u \otimes (\partial_B v)$$

for all $u \in A_p$ and $v \in B_q$. It is well-known that $\partial_C^2 = 0$ so that C_* with ∂_C is indeed a chain complex.
Furthermore, by a theorem of Künneth, we have the following relation for homologies:

$$H_*\,(C_*) \cong H_*\,(A_*) \otimes H_*\,(B_*) \tag{2.39}$$

that is,

$$H_r\,(C_*) \cong \bigoplus_{\{p,q\ge 0:\, p+q=r\}} H_p\,(A_*) \otimes H_q\,(B_*)\,.$$

Proof of Theorem 2.3. Let us restate (2.7) in equivalent form as follows: for any $r \geq 0$,

$$\Omega_{r-1}(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} \Omega_{p-1}(X) \otimes \Omega_{q-1}(Y).$$

Using the notation

$$\Omega_p' = \Omega_{p-1},$$

we rewrite the above identity in the form

$$\Omega_r'(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} \Omega_p'(X) \otimes \Omega_q'(Y). \tag{2.40}$$

On any digraph we consider the chain complex

$$\Omega_*' = \{\Omega_p'\}_{p \geq 0}$$

as well as $\mathcal{R}_*' = \{\mathcal{R}_p'\}_{p \geq 0}$ where $\mathcal{R}_p' = \mathcal{R}_{p-1}$. In fact, we will prove that

$$\Omega_*'(Z) \cong \Omega_*'(X) \otimes \Omega_*'(Z), \tag{2.41}$$

which contains (2.40).
It follows from the product rule (2.5) that, for all $u \in \mathcal{R}_p'(X)$ and $v \in \mathcal{R}_q'(Y)$,

$$\partial(u * v) = (\partial u) * v + (-1)^p u * \partial v. \tag{2.42}$$

Consider the tensor product

$$\mathcal{R}_*(X, Y) := \mathcal{R}_*'(X) \otimes \mathcal{R}_*'(Y),$$

that is, for any $r \geq 0$,

$$\mathcal{R}_r(X, Y) = \bigoplus_{\{p,q \geq 0:\ p+q=r\}} \mathcal{R}_p'(X) \otimes \mathcal{R}_q'(Y).$$

By the definition of the operator ∂ on $\mathcal{R}_*(X, Y)$, we have, for all $u \in \mathcal{R}_p'(X)$ and $v \in \mathcal{R}_q'(Y)$,

$$\partial(u \otimes v) = (\partial u) \otimes v + (-1)^p u \otimes (\partial v). \tag{2.43}$$

Consider also a linear mapping

$$\Phi : \mathcal{R}_r(X, Y) \to \mathcal{R}_r'(Z)$$

that is defined on the basis elements by

$$\Phi(e_x \otimes e_y) = e_x * e_y,$$

for all regular elementary $(p-1)$-paths e_x on X and $(q-1)$-paths e_y on Y with $p + q = r$.
It follows that, for all $u \in \mathcal{R}_p'(X)$ and $v \in \mathcal{R}_q'(Y)$,

$$\Phi(u \otimes v) = u * v.$$

Comparison of (2.42) and (2.43) shows that Φ commutes with ∂, as

$$\begin{aligned}\Phi\left(\partial\left(u\otimes v\right)\right) &= \Phi\left(\partial u\otimes v+\left(-1\right)^{p}u\otimes\partial v\right)=\partial u\ast v+\left(-1\right)^{p}u\ast\partial v\\ &= \partial\left(u\ast v\right)=\partial\Phi\left(u\otimes v\right).\end{aligned}$$

Hence, Φ is a homomorphism of the chain complexes $\mathcal{R}_{\ast}\left(X,Y\right)$ and $\mathcal{R}_{\ast}^{\prime}\left(Z\right)$.

Let us show that Φ is in fact a monomorphism. Indeed, the basis in $\mathcal{R}_{r}\left(X,Y\right)$ consists of all the elements of the form $e_{x}\otimes e_{y}$, for all regular elementary $\left(p-1\right)$-paths e_{x} on X and $\left(q-1\right)$-paths e_{y} on Y with $p+q=r$. Since $\Phi\left(e_{x}\otimes e_{y}\right)=e_{x}\ast e_{y}$, and all the paths $e_{x}\ast e_{y}$ are linearly independent in $\mathcal{R}_{r}^{\prime}\left(Z\right)$, we conclude that Φ is a monomorphism.

Consider now the chain complex

$$\Omega_{\ast}\left(X,Y\right):=\Omega_{\ast}^{\prime}\left(X\right)\otimes\Omega_{\ast}^{\prime}\left(Y\right),$$

that is, for any $r\geq 0$,

$$\Omega_{r}\left(X,Y\right)=\bigoplus_{\{p,q\geq 0:\ p+q=r\}}\Omega_{p}^{\prime}\left(X\right)\otimes\Omega_{q}^{\prime}\left(Y\right).$$

It follows from the definition of Φ that

$$\Phi\left(\Omega_{r}\left(X,Y\right)\right)=\sum_{\{p,q\geq 0:\,p+q+1=r\}}\Omega_{p}^{\prime}\left(X\right)\ast\Omega_{q}^{\prime}\left(Y\right). \tag{2.44}$$

However, by Theorem 2.7 the right hand side of (2.44) coincides with $\Omega_{r}^{\prime}\left(Z\right)$. Therefore, Φ is an isomorphism of the linear spaces $\Omega_{r}\left(X,Y\right)$ and $\Omega_{r}^{\prime}\left(Z\right)$ and, hence, an isomorphism of the chain complexes $\Omega_{\ast}\left(X,Y\right)$ and $\Omega_{\ast}^{\prime}\left(Z\right)$, which proves (2.41).

Finally, (2.8) follows from (2.41) and the Künneth theorem (2.39), and (2.9) is an obvious consequence of (2.8). ■

3 Cartesian product of digraphs

In this this section we use the chain complexes

$$\{0\} \xleftarrow{\partial} \mathcal{R}_0 \xleftarrow{\partial} \mathcal{R}_1 \xleftarrow{\partial} \ldots$$

and

$$\{0\} \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \ldots.$$

3.1 Cross product of paths

Given two finite sets X, Y, consider their product

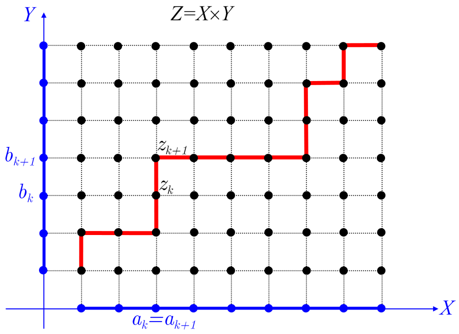
$$Z = X \times Y = \{(a, b) : a \in X \text{ and } b \in Y\}.$$

Definition. Let $z = z_0 z_1 \ldots z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 0, \ldots, r-1$,

$$\text{either } a_k = a_{k+1} \text{ or } b_k = b_{k+1}.$$

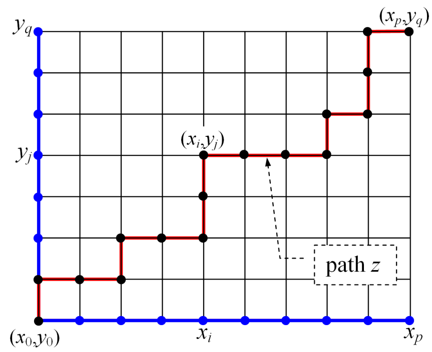
If $a_k = a_{k+1}$ then the couple $z_k z_{k+1}$ is called *vertical*. This case is shown on this diagram:

If $b_k = b_{k+1}$ then $z_k z_{k+1}$ is called *horizontal*.



Given a stair-like path $z = z_0 \ldots z_r$ on Z where $z_k = (a_k, b_k)$, define its projection onto X as an elementary path $x = \{x_i\}$ on X that is obtained from the path $a_0 \ldots a_r$ by collapsing any sequence of subsequent repeated vertices into one vertex.

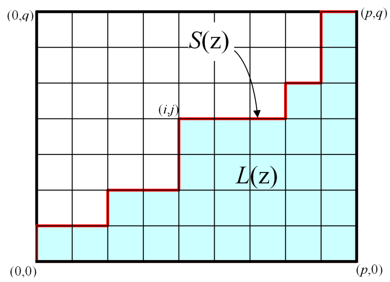
In the same way define the projection of z onto Y and denote it by $y = \{y_j\}$.



The projections $x = x_0 \ldots x_p$ and $y = y_0 \ldots y_q$ are regular elementary paths, and $p + q = r$.

Every vertex (x_i, y_j) of the path z can be represented by a point (i, j) of $\mathbb{Z}_+^2$ so that the path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}_+^2$ connecting $(0,0)$ and (p, q).

Define the *elevation* $L(z)$ of z as the number of cells in $\mathbb{Z}_+^2$ below the staircase $S(z)$.



Definition. For given regular elementary paths x on X and y on Y, denote by $\Pi_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are equal to x and y, respectively.

Definition. Define the *cross product* of the paths $e_x \in R(X)$ and $e_y \in R(Y)$ as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z \tag{3.1}$$

and extend it by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

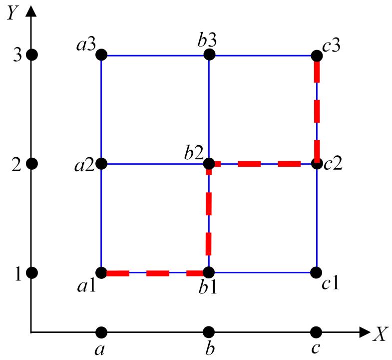
Example. Let us denote the vertices on X by letters a, b, c etc. and the vertices on Y by integers $1, 2, 3$, etc. so that the vertices on Z can be denoted as $a1, b2$ etc. as chessboard fields. Then we have

$$e_a \times e_{12} = e_{a1\,a2}, \quad e_{ab} \times e_1 = e_{a1\,b1}$$

$$e_{ab} \times e_{12} = e_{a1\,b1\,b2} - e_{a1\,a2\,b2}$$

$$e_{ab} \times e_{123} = e_{a1\,b1\,b2\,b3} - e_{a1\,a2\,b2\,b3} + e_{a1\,a2\,a3\,b3}$$

$$\begin{aligned} e_{abc} \times e_{123} = {} & e_{a1\,b1\,c1\,c2\,c3} - e_{a1\,b1\,b2\,c2\,c3} + e_{a1\,b1\,b2\,b3\,c3} \\ & + e_{a1\,a2\,b2\,c2\,c3} - e_{a1\,a2\,b2\,b3\,c3} + e_{a1\,a2\,a3\,b3\,c3} \end{aligned}$$



Let us state some properties of the cross product.

Lemma 3.1. *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = \partial u \times v + (-1)^p u \times \partial v. \tag{3.2}$$

Proof. It suffices to prove (3.2) for the case $u = e_x$ and $v = e_y$ where

$$x = x_0 \dots x_p \ \text{ and } \ y = y_0 \dots y_q$$

are regular elementary p-path on X and q-path on Y, respectively. Set $r = p + q$ so that $e_x \times e_y \in \mathcal{R}_r(Z)$.

If $p = q = 0$ then all the terms in (3.2) vanish. Assume $p = 0$ and $q \geq 1$ (the case $p \geq 1$ and $q = 0$ is similar). Then $\Pi_{x,y}$ contains the only element $z = z_0 \dots z_r$ where $z_i = (x_0, y_i)$. Since $L(z) = 0$, we obtain by (3.1) that

$$e_x \times e_y = e_z.$$

By (1.1) have

$$\partial(e_x \times e_y) = \partial e_{z_0 \dots z_r} = \sum_{k=0}^{r} (-1)^k e_{z_{(k)}},$$

where we use the notation

$$z_{(k)} = z_0 \dots \widehat{z_k} \dots z_r = z_0 \dots z_{k-1} z_{k+1} \dots z_r.$$

Since $e_{z_{(k)}} = e_x \times e_{y_{(k)}}$ and $r = q$, it follows that

$$\partial(e_x \times e_y) = \sum_{k=0}^{q} (-1)^k e_x \times e_{y_{(k)}} = e_x \times \partial e_y = u \times \partial v,$$

which implies (3.2), because $\partial u = 0$.
Consider now the main case when $p, q \geq 1$. We have by (3.1) and (1.1),

$$\partial \left(e_x \times e_y\right) = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \partial e_z = \sum_{z \in \Pi_{x,y}} \sum_{k=0}^{r} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{3.3}$$

Switching the order of the sums, rewrite (3.3) in the form

$$\partial \left(e_x \times e_y\right) = \sum_{k=0}^{r} \sum_{z \in \Pi_{x,y}} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{3.4}$$

Given an index $k = 0, ..., r$ and a path $z \in \Pi_{x,y}$, consider the following four logically possible cases of horizontal and vertical couples around z_k:

$$(H):\quad \overset{z_{k-1}}{\bullet} \;-\; \overset{z_k}{\bullet} \;-\; \overset{z_{k+1}}{\bullet} \qquad\qquad (V):\quad \begin{array}{c} \overset{z_{k+1}}{\bullet} \\ | \\ \overset{z_k}{\bullet} \\ | \\ \overset{z_{k-1}}{\bullet} \end{array}$$

$$(R):\quad \begin{array}{ccc} & & \overset{z_{k+1}}{\bullet} \\ & & | \\ \overset{z_{k-1}}{\bullet} & - & \overset{z_k}{\bullet} \end{array} \qquad\qquad (L):\quad \begin{array}{ccc} \overset{z_k}{\bullet} & - & \overset{z_{k+1}}{\bullet} \\ | & & \\ \overset{z_{k-1}}{\bullet} & & \end{array}$$

Here (H) stands for a horizontal position, (V) for vertical, (R) for right and (L) for left. If $k = 0$ or $k = r$ then z_{k-1} or z_{k+1} should be ignored, so that one has only two distinct positions (H) and (V).
If $z \in \Pi_{x,y}$ and z_k is in position (R) or (L) then consider a path $z' \in \Pi_{x,y}$ such that $z'_i = z_i$ for all $i \neq k$, whereas z'_k is in the opposite position (L) or (R), respectively, as on the diagrams:

$$\begin{array}{ccc} \overset{z'_k}{\bullet} & - & \overset{z_{k+1}}{\bullet} \\ | & & | \\ \overset{z_{k-1}}{\bullet} & - & \overset{z_k}{\bullet} \end{array} \qquad\qquad \begin{array}{ccc} \overset{z_k}{\bullet} & - & \overset{z_{k+1}}{\bullet} \\ | & & | \\ \overset{z_{k-1}}{\bullet} & - & \overset{z'_k}{\bullet} \end{array}$$

Clearly, we have $L(z') = L(z) \pm 1$ and $e_{z_{(k)}} = e_{z'_{(k)}}$, which implies that the terms $e_{z_{(k)}}$ and $e_{z'_{(k)}}$ in (3.4) cancel out. Hence, in the interior sum in (3.4), all the terms $e_{z_{(k)}}$ with z_k in positions (R) and (L) cancel out.
Denote by $\Pi^k_{x,y}$ the set of paths $z \in \Pi_{x,y}$ such that z_k is in position (V) and by ${\Pi_{x,y}}^{k}$ the set of paths $z \in \Pi_{x,y}$ such that z_k is in position (H). By the above observation, we can restrict the summation in (3.4) to those pairs k, z where z_k is in positions (V) or (H), that is,

$$\partial \left(e_x \times e_y\right) = \sum_{k=0}^{r} \sum_{z \in \Pi^k_{x,y} \sqcup {\Pi_{x,y}}^{k}} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{3.5}$$

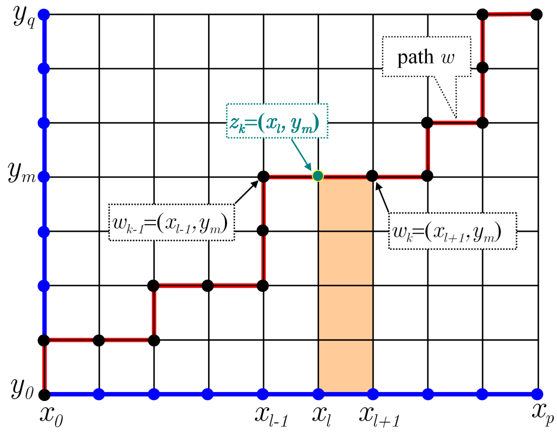
Let us now compute the first term in the right hand side of (3.2):

$$\partial e_x \times e_y = \sum_{l=0}^{p} (-1)^l e_{x_{(l)}} \times e_y = \sum_{l=0}^{p} \sum_{w \in \Pi_{x_{(l)},y}} (-1)^{L(w)+l} e_w. \tag{3.6}$$

Fix some $0 \le l \le p$ and $w \in \Pi_{x_{(l)},y}$. Since the projection of w on X is

$$x_{(l)} = x_0 ... x_{l-1} x_{l+1} ... x_p,$$

there exists a unique index k such that w_{k-1} projects onto x_{l-1} and w_k projects onto x_{l+1}. Then w_{k-1} and w_k project on the same vertex of Y, say y_m.



Paths w and z.

The shaded area represents $L(z) - L(w)$.

Define a path $z \in \Pi_{x,y}^k$ by adding to w one vertex (x_k, y_m) between w_{k-1} and w_k as follows:

$$z_i = \begin{cases} w_i & \text{for } i \le k-1, \\ (x_l, y_m) & \text{for } i = k, \\ w_{i-1} & \text{for } i \ge k+1. \end{cases} \tag{3.7}$$

By construction we have $z_{(k)} = w$. It also follows from the construction that

$$L(z) = L(w) + m.$$

Since $k = l + m$, we obtain that

$$L(z) + k = L(w) + l + 2m.$$

We see that each pair l, w where $l = 0, ..., p$ and $w \in \Pi_{x_{(l)},y}$ gives rise to a pair k, z where $k = 0, ..., r$, $z \in \Pi_{x,y}^k$, and

$$(-1)^{L(z)+k} e_{z_{(k)}} = (-1)^{L(w)+l} e_w.$$

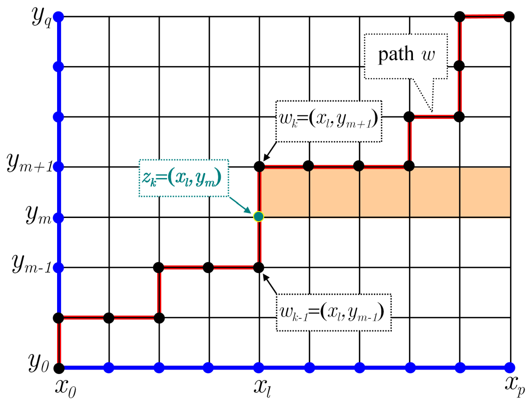
By reversing this argument, we obtain that each such pair k, z gives back l, w so that this correspondence between k, z and l, w is bijective. Hence, we conclude that

$$\partial e_x \times e_y = \sum_{l=0}^{p} \sum_{w \in \Pi_{x_{(l)},y}} (-1)^{L(w)+l} e_w = \sum_{k=0}^{r} \sum_{z \in \Pi_{x,y}^k} (-1)^{L(z)+k} e_{z_{(k)}}. \tag{3.8}$$

In the same way we will handle the second term in the right hand side of (3.2). First we have

$$(-1)^p e_x \times \partial e_y = \sum_{m=0}^{q} (-1)^{m+p} e_x \times e_{y_{(m)}} = \sum_{m=0}^{q} \sum_{w \in \Pi_{x,y_{(m)}}} (-1)^{L(w)+m+p} e_w.$$

Each pair m, w here gives rise to a pair k, z, where $k = 0, ..., r$ and $z \in \Pi^k_{x,y}$, as follows: choose k such that w_{k-1} projects onto y_{m-1} and w_k projects onto y_{m+1}.
Then w_{k-1} and w_k have the same projection onto X, say x_l.



Paths w and z.

The shaded area represents $L(z) - L(w)$.

Define the path $z \in \Pi^k_{x,y}$ as in (3.7), that is, by adding to w the vertex (x_l, y_m) between w_{k-1} and w_k. Then we have $w = z_{(k)}$ and

$$L(z) = L(w) + p - l.$$

Since $k = l + m$, we obtain

$$L(z) + k = L(w) + p + m$$

and

$$(-1)^p \, e_x \times \partial e_y = \sum_{m=0}^{q} \sum_{w \in \Pi_{x,y_{(m)}}} (-1)^{L(w)+m+p} \, e_w = \sum_{k=0}^{r} \sum_{z \in \Pi^k_{x,y}} (-1)^{L(z)+k} \, e_{z_{(k)}} \, .$$

Combining this with (3.5) and (3.8), we obtain (3.2). ■

Lemma 3.2. *If* $u \in \mathcal{R}_p(X), \varphi \in \mathcal{R}_{p'}(X)$ *and* $v \in \mathcal{R}_q(Y), \psi \in \mathcal{R}_{q'}(Y)$, *where* $p, q, p', q' \geq 0$, *then*

$$\langle u \times v, \varphi \times \psi \rangle = \binom{p+q}{p} \langle u, \varphi \rangle \langle v, \psi \rangle \, . \tag{3.9}$$

Proof. It from (3.1) that

$$\begin{aligned} u \times v &= \sum_{x \in R(X)} \sum_{y \in R(Y)} u^x v^y \, (e_x \times e_y) \\ &= \sum_{x \in R(X)} \sum_{y \in R(Y)} u^x v^y \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \, e_z \\ &= \sum_{z \in S(Z)} (-1)^{L(z)} u^x v^y e_z, \end{aligned}$$

where $S(Z)$ denotes the set of all stair-like paths on Z, and x, y are the projections of z onto Z and Y, respectively. It follows that, for any $z \in S(Z)$,

$$(u \times v)^z = (-1)^{L(z)} \, u^x v^y. \tag{3.10}$$

Similarly, we have

$$(\varphi \times \psi)^z = (-1)^{L(z)} \, \varphi^x \psi^y,$$

whence it follows that

$$\langle u \times v, \varphi \times \psi \rangle = \sum_{z \in R(\mathbb{Z})} (u \times v)^z \, (\varphi \times \psi)^z$$

$$
\begin{aligned}
&= \sum_{z\in S(Z)} (-1)^{L(z)}\, u^x v^y\, (-1)^{L(z)}\, \varphi^x \psi^y \quad (x, y \text{ are projections of } z) \\
&= \sum_{x\in R(X)} \sum_{y\in R(Y)} \sum_{z\in \Pi_{x,y}} u^x v^y \varphi^x \psi^y \\
&= \sum_{x\in R_p(X)} \sum_{y\in R_q(Y)} \sum_{z\in \Pi_{x,y}} u^x v^y \varphi^x \psi^y .
\end{aligned}
$$

Here we have restricted the summation to $x \in R_p(X)$ and $y \in R_q(Y)$ because $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$. Since the summand does not depend on z, we obtain

$$
\langle u \times v, \varphi \times \psi\rangle = \sum_{x\in R_q(X)} \sum_{y\in R_q(Y)} |\Pi_{x,y}|\, u^x \varphi^x v^y \psi^y = \tbinom{p+q}{p} \langle u, \varphi\rangle \langle v, \psi\rangle
$$

where we have used that

$$
|\Pi_{x,y}| = \binom{p+q}{p}. \tag{3.11}
$$

Indeed, every path $z = z_0...z_r \in \Pi_{x.y}$ is uniquely determined by the choice of p vertices z_k out of r vertices $z_0...z_{r-1}$ such that the pair $z_k z_{k+1}$ is horizontal, which implies (3.11). ■

3.2 Paths on Cartesian products of digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define their Cartesian product as a digraph $Z = X\Box Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;
- the edges in Z are of two types: $(a, b) \to (a', b)$ where $a \to a'$ (a *horizontal* edge) and $(a, b) \to (a, b')$ where $b \to b'$ (a *vertical* edge):

$$
\begin{array}{ccccccc}
b'\bullet & \dots & \overset{(a,b')}{\bullet} & \to & \overset{(a',b')}{\bullet} & \dots \\
\uparrow & & \uparrow & & \uparrow & \\
b\bullet & \dots & \overset{(a,b)}{\bullet} & \to & \overset{(a',b)}{\bullet} & \dots \\
& & & & & \\
{}^{Y}\diagup_{X} & \dots & \underset{a}{\bullet} & \to & \underset{a'}{\bullet} & \dots
\end{array}
$$

It follows that any allowed elementary path in Z is stair-like. Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed. If $e_x \in A_p(X)$ and $e_y \in A_q(Y)$ then $e_x \times e_y \in \mathcal{A}_{p+q}(Z)$ because by (3.1)

$$
e_x \times e_y = \sum_{z\in \Pi_{x,y}} (-1)^{L(z)}\, e_z
$$

and any $z \in \Pi_{x,y}$ belongs to $A_{p+q}(Z)$. Consequently, we have

$$
u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \;\Rightarrow\; u \times v \in \mathcal{A}_{p+q}(Z). \tag{3.12}
$$

Lemma 3.3. *Let $r = p + q$ where $p, q \geq 0$.*
(a) *We have*

$$u \in \Omega_p(X) \ \text{ and } \ v \in \Omega_q(Y) \Rightarrow u \times v \in \Omega_r(Z).$$

(b) *The cross product is well defined for homology classes:*

$$u \in H_p(X) \ \text{ and } \ v \in H_q(Y) \Rightarrow u \times v \in H_r(Z).$$

The proof is based on the product rule (3.2).

Lemma 3.4. *Any path $w \in \Omega_*(Z)$ admits a representation*

$$w = \sum_{x \in A(X),\ y \in A(Y)} c^{xy} (e_x \times e_y) \tag{3.13}$$

with some coefficients $c^{xy} \in \mathbb{K}$ (only finitely many coefficients are non-vanishing). Furthermore, the coefficients c^{xy} are uniquely determined by w.

Remark. Note that the representation (3.13) is fails in general for $w \in \mathcal{A}_*(Z)$ (unlike the case of join). Indeed, if $X = \{a \to b\}$ and $Y = \{0 \to 1\}$ then there is only one path $e_x \in \mathcal{A}_1(X)$ that is e_{ab}, and only one path $e_y \in \mathcal{A}_1(Y)$ that is e_{01}. There cross product is the ∂-invariant path

$$w = e_{a0\,b0\,b1} - e_{a0\,a1\,b1} \in \Omega_2(Z).$$

Clearly, the allowed path $e_{a0\,b0\,b1} \in \mathcal{A}_2(Z)$ does not admit representation (3.13).

Proof. Let us first show the uniqueness of c^{xy}, which is equivalent to the linear independence of the family $\{e_x \times e_y\}$ across all $x \in A(X)$ and $y \in A(Y)$. Indeed, assume that, for some scalars c^{xy},

$$\sum_{x \in A(X), y \in A(Y)} c^{xy} e_x \times e_y = 0,$$

and prove that $c^{xy} = 0$ for any couple x, y as in the summation. Indeed, by (3.1) we have

$$\begin{aligned}\sum_{x \in A(X), y \in A(Y)} c^{xy} e_x \times e_y &= \sum_{x \in A(X), y \in A(Y)} \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} c_{xy} e_z \\ &= \sum_{z \in A(Z)} (-1)^{L(z)} c_{xy} e_z,\end{aligned}$$

where the summation is taken over all allowed (and, hence, stair-like) paths z on Z, and x, y are the projections of z onto X and Y, respectively. Since all the paths e_z in this sum are distinct and, hence, linearly independent, we obtain $c_{xy} = 0$.

For the second part of the proof, recall that, by (1.1), for any p-path $e_{i_0 \ldots i_p}$ on any finite set V, we have

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \widehat{i_q} \ldots i_p}.$$

This formula implies the following: for any p-path

$$v = \sum v^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$$

on V and any elementary $(p-1)$-path $i_0...i_p$, the coefficient $(\partial v)^{i_0...i_{p-1}}$ of ∂v is given by

$$(\partial v)^{i_0...i_{p-1}} = \sum_{j\in V}\sum_{q=0}^{p} (-1)^q \, v^{i_0...i_{q-1}j\, i_q...i_{p-1}}. \tag{3.14}$$

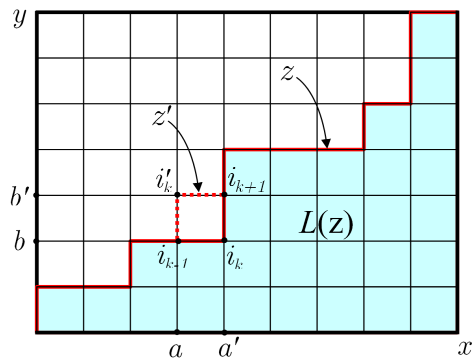
Indeed, the elementary term $e_{i_0...i_{p-1}}$ can appear in ∂v from any elementary path of v of the form $e_{i_0...i_{q-1}j i_q...i_{p-1}}$ where j is an arbitrary vertex at an arbitrary position q. Summing up in j and q all the coefficients $v^{i_0...i_{q-1}j\, i_q...i_{p-1}}$ of such paths, we obtain (3.14).

Now let us prove the existence of representation (3.13) for any $w\in\Omega_r(Z)$ with any $r\geq 0$. For any $x\in A(X)$ and $y\in A(Y)$, define the coefficient c_{xy} by the formula

$$c^{xy} = (-1)^{L(z)}\, w^z, \tag{3.15}$$

where $z\in\Pi_{x,y}$ is arbitrary. Let us verify that the value of c^{xy} in (3.15) is independent of the choice of $z\in\Pi_{x,y}$. Set $z=i_0...i_r$. Let k be an index such that one of the couples $i_{k-1}i_k$, i_ki_{k+1} is vertical and the other is horizontal. If $i_{k-1}=(a,b)$ and $i_{k+1}=(a',b')$ where $a,a'\in X$ and $b,b'\in Y$, then i_k is either (a',b) or (a,b').

Denote the other of these two vertices by i_k' as, for example, on the following diagram: Replacing in $z=i_0...i_r$ the vertex i_k by i_k', we obtain the path $z'=i_0...i_{k-1}i_k'i_{k+1}...i_r$ that also belongs to $\Pi_{x,y}$ (and is allowed). Since the $(r-1)$-path $i_0...i_{k-1}i_{k+1}...i_r$ is regular but non-allowed (as $i_{k-1}\not\to i_{k+1}$), we have



$$(\partial w)^{i_0...i_{k-1}i_{k+1}...i_r} = 0. \tag{3.16}$$

On the other hand, we have by (3.14)

$$(\partial w)^{i_0...i_{k-1}i_{k+1}...i_r} = \sum_{j\in Z}\left(\sum_{q=0}^{k-1}(-1)^q\, w^{i_0...i_{q-1}ji_q...\underline{i_{k-1}i_{k+1}}...i_r}\right. \tag{3.17}$$

$$+ (-1)^k\, w^{i_0...\underline{i_{k-1}ji_{k+1}}...i_r} \tag{3.18}$$

$$\left.+\sum_{q=k+2}^{r+1}(-1)^{q-1}\, w^{i_0...\underline{i_{k-1}i_{k+1}}...i_{q-1}ji_q...i_r}\right). \tag{3.19}$$

All the components of w in the sums (3.17) and (3.19) correspond to elementary paths on Z containing consecutive vertices i_{k-1} and i_{k+1}. Since $i_{k-1}\not\to i_{k+1}$ in Z, all these elementary paths are non-allowed. Since w is allowed, all its components in the sums (3.17) and (3.19) vanish. The path $i_0...i_{k-1}ji_{k+1}...i_r$ in the term (3.18) is also non-allowed unless $j=i_k$ or $j=i_k'$ (note that i_k and i_k' are uniquely determined by i_{k-1} and i_{k+1}). Hence, the only non-zero terms in (3.18) are

$$w^{i_0...i_{k-1}i_ki_{k+1}...i_r} = w^z \text{ and } w^{i_0...i_{k-1}i_k'i_{k+1}...i_r} = w^{z'}.$$

Combining (3.16) and (3.17)-(3.19), we obtain

$$0 = w^z + w^{z'}.$$

Since $L(z') = L(z) \pm 1$, it follows that

$$(-1)^{L(z')} w^{z'} = (-1)^{L(z)} w^z. \tag{3.20}$$

The transformation $z \mapsto z'$ described above, allows us to obtain from a given path $z \in \Pi_{x,y}$ any other path in $\Pi_{x,y}$ in a finite number of steps. Since the quantity $(-1)^{L(z)} w^z$ does not change under this transformation, it follows that it does not depend on a particular choice of $z \in \Pi_{x,y}$, which was claimed. Hence, the coefficients c^{xy} are well-defined by (3.15).

Finally, let us show that the equality (3.13) holds with the coefficients c^{xy} from (3.15). By (3.1) we have

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z.$$

Using (3.15) we obtain

$$\begin{aligned}
\sum_{x \in A(X),\ y \in A(Y)} c^{xy} (e_x \times e_y) &= \sum_{x \in A(X),\ y \in A(Y)} c^{xy} \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z \\
&= \sum_{x \in A(X),\ y \in A(Y)} \sum_{z \in \Pi_{x,y}} w^z e_z \\
&= \sum_{z \in A(Z)} w^z e_z = w,
\end{aligned}$$

which finishes the proof. ■

Lemma 3.5. *Any path $w \in \Omega_* (Z)$ admits representations*

$$w = \sum_{x \in A(X)} e_x \times a^x = \sum_{y \in A(Y)} b^y \times e_y \tag{3.21}$$

where $a^x \in \Omega_* (Y)$ and $b^y \in \Omega_* (X)$ are uniquely determined.

Lemma 3.6. *Let $r = p + q$. If*

$$u \in \Omega_p^{\perp} (X) \ \ \textit{and} \ \ v \in \mathcal{A}_q (Y) \Rightarrow \ u \times v \in \Omega_r^{\perp} (Z)$$

and

$$u \in \mathcal{A}_p (X) \ \ \textit{and} \ \ v \in \Omega_q^{\perp} (Y) \Rightarrow u \times v \in \Omega_r^{\perp} (Z) .$$

Theorem 3.7. *Any path $w \in \Omega_r (Z)$ with $r \geq 0$ admits a representation of the form*

$$w = \sum_{i=1}^{k} u_i \times v_i$$

for some finite k, with some $u_i \in \Omega_{p_i}(X)$ and $v_i \in \Omega_{q_i}(Y)$, where $p_i, q_i \geq 0$ and $p_i + q_i = r$.

3.3 Künneth formula for product

Here is the main result of this section. We use here the path chain complexes $\Omega_* = \{\Omega_p\}_{p\geq 0}$ and their homologies $\{H_p\}_{p\geq 0}$.

Theorem 3.8. (Künneth formula for product) *Let X, Y be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of chain complexes:*

$$\Omega_*(Z) \cong \Omega_*(X) \otimes \Omega_*(Y),$$

that is given by $u \otimes v \mapsto u \times v$ for $u \in \Omega_*(X)$ and $v \in \Omega_*(Y)$. In particular, for any $r \geq 0$,

$$\Omega_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} \Omega_p(X) \otimes \Omega_q(Y). \tag{3.22}$$

Consequently, we have

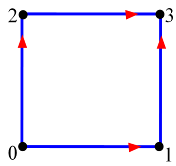
$$H_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} H_p(X) \otimes H_q(Y), \tag{3.23}$$

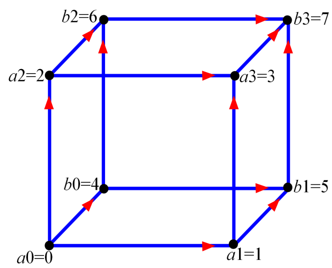
$$|\Omega_r(Z)| = \sum_{\{p,q\geq 0: p+q=r\}} |\Omega_p(X)| \, |\Omega_q(Y)|, \tag{3.24}$$

and

$$\beta_r(Z) = \sum_{\{p,q\geq 0: p+q=r\}} \beta_p(X) \, \beta_q(Y). \tag{3.25}$$

Example. Let X be an interval and Y be a square:

$X = {}^a\bullet \to \bullet^b$ and $Y =$ 



Then $Z = X \Box Y$ is a 3-cube:

We have:

$\Omega_1(X) = \langle e_{ab} \rangle$, $\Omega_p(X) = 0$ for $p \geq 2$,

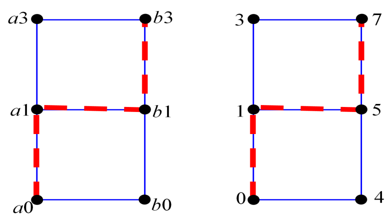
$\Omega_1(Y) = \langle e_{01}, e_{13}, e_{23}, e_{02} \rangle$,

$\Omega_2(Y) = \langle e_{013} - e_{023} \rangle$, $\Omega_q(Y) = 0$ for $q \geq 3$.

By (3.22) we obtain

$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) = \langle e_{ab} \times (e_{013} - e_{023}) \rangle.$$

Let us compute the cross-products:



$$\begin{aligned} e_{ab} \times e_{013} &= e_{a0\,b0\,b1\,b3} - e_{a0\,a1\,b1\,b3} + e_{a0\,a1\,a3\,b3} \\ &= e_{0457} - e_{0157} + e_{0137} \end{aligned}$$

and

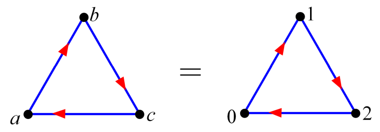
$$e_{ab} \times e_{023} = e_{0467} - e_{0267} + e_{0237}.$$

Hence, we obtain

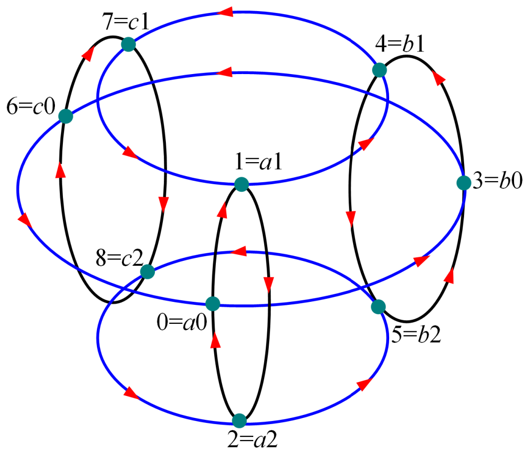
$$\Omega_3(Z) = \langle e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237} \rangle .$$

That is, Ω_3 is generated by a single ∂-invariant 3-path that is associated with the 3-cube.

Example. Denote by T the following 3-cycle (=1-torus):

$T =$ 

Consider the 2-torus $T^2 = T \Box T$ that is shown here:



Let us compute $\Omega_r(T^2)$ and $H_r(T^2)$.
We have

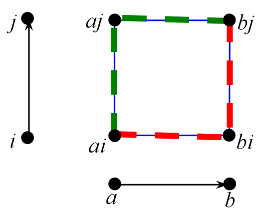
$$\Omega_0(T) = \langle e_0, e_1, e_2 \rangle,$$
$$\Omega_1(T) = \langle e_{01}, e_{12}, e_{20} \rangle,$$
$$\Omega_p(T) = \{0\} \text{ for } p \geq 2.$$

By (3.22) we obtain $\Omega_r(T^2) = \{0\}$ for $r \geq 3$ and

$$\begin{aligned}\Omega_2\left(T^2\right) &= \Omega_1(T) \otimes \Omega_1(T) = \langle e_{ab}, e_{bc}, e_{ca} \rangle \times \langle e_{01}, e_{12}, e_{20} \rangle \\ &= \langle e_{ab} \times e_{01},\ e_{ab} \times e_{12},\ e_{ab} \times e_{20},\ e_{bc} \times e_{01},\ e_{bc} \times e_{12}, \\ &\quad e_{bc} \times e_{20},\ e_{ca} \times e_{01},\ e_{ca} \times e_{12},\ e_{ca} \times e_{20} \rangle .\end{aligned}$$

Using that

$$e_{ab} \times e_{ij} = e_{ai\,bi\,bj} - e_{ai\,aj\,bj}$$



we obtain that

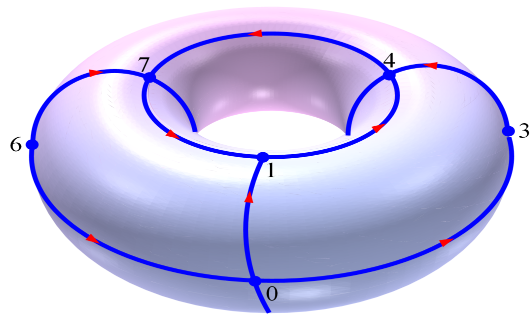
$$\begin{aligned}\Omega_2\left(T^2\right) = \langle & e_{a0\,b0\,b1} - e_{a0\,a1\,b1},\ e_{a1\,b1\,b2} - e_{a1\,a2\,b2},\ e_{a2\,b2\,b0} - e_{a2\,a0\,b0}, \\ & e_{b0\,c0\,c1} - e_{b0\,b1\,c1},\ e_{b1\,c1\,c2} - e_{b1\,b2\,c2},\ e_{b2\,c2\,c0} - e_{b2\,b0\,c0}, \\ & e_{c0\,a0\,a1} - e_{c0\,c1\,a1},\ e_{c1\,a1\,a2} - e_{c1\,c2\,a2},\ e_{c2\,a2\,a0} - e_{c2\,c0\,a0} \rangle .\end{aligned}$$

That is, using a simple numbering of the vertices of T^2, we have

$$\begin{aligned}\Omega_2\left(T^2\right) = \langle & e_{034} - e_{014},\ e_{145} - e_{125},\ e_{253} - e_{203}, \\ & e_{367} - e_{347},\ e_{478} - e_{458},\ e_{586} - e_{536} \\ & e_{601} - e_{671},\ e_{712} - e_{782},\ e_{820} - e_{860} \rangle .\end{aligned} \tag{3.26}$$

Hence, $\Omega_2\left(T^2\right)$ is generated by 9 squares.

This can be visualized using the following embedding of T^2 onto a topological torus:



Let us compute the homology groups of T^2. We know that

$$H_0\left(T\right)=\langle e_0\rangle,\quad H_1\left(T\right)=\langle e_{01}+e_{12}+e_{20}\rangle,\quad H_p\left(T\right)=\{0\}\ \text{for}\ p\geq 2.$$

By (3.23) we obtain

$$H_1\left(T^2\right)=H_0\left(T\right)\otimes H_1\left(T\right)+H_1\left(T\right)\otimes H_0\left(T\right)=\langle v_1,v_2\rangle$$

where

$$\begin{aligned}v_1&=e_a\times\left(e_{01}+e_{12}+e_{20}\right)=e_{a0\,a1}+e_{a1\,a2}+e_{a2\,a0}=e_{01}+e_{12}+e_{20}\\ v_2&=\left(e_{ab}+e_{bc}+e_{ca}\right)\times e_0=e_{a0\,b0}+e_{b0\,c0}+e_{c0\,a0}=e_{03}+e_{36}+e_{60}.\end{aligned}$$

Again by (3.23) we get

$$H_2\left(T^2\right)=H_1\left(T\right)\otimes H_1\left(T\right)=\langle u\rangle,$$

where

$$u=\left(e_{ab}+e_{bc}+e_{ca}\right)\times\left(e_{01}+e_{12}+e_{20}\right),$$

Hence,

$$\begin{aligned}u=\;&e_{a0\,b0\,b1}-e_{a0\,a1\,b1}+e_{a1\,b1\,b2}-e_{a1\,a2\,b2}+e_{a2\,b2\,b0}-e_{a2\,a0\,b0}\\ &+e_{b0\,c0\,c1}-e_{b0\,b1\,c1}+e_{b1\,c1\,c2}-e_{b1\,b2\,c2}+e_{b2\,c2\,c0}-e_{b2\,b0\,c0}\\ &+e_{c0\,a0\,a1}-e_{c0\,c1\,a1}+e_{c1\,a1\,a2}-e_{c1\,c2\,a2}+e_{c2\,a2\,a0}-e_{c2\,c0\,a0},\end{aligned}$$

that is,

$$\begin{aligned}u=\;&\left(e_{034}-e_{014}\right)+\left(e_{145}-e_{125}\right)+\left(e_{253}-e_{203}\right)+\left(e_{367}-e_{347}\right)+\left(e_{478}-e_{458}\right)\\ &+\left(e_{586}-e_{536}\right)+\left(e_{601}-e_{671}\right)+\left(e_{712}-e_{782}\right)+\left(e_{820}-e_{860}\right).\end{aligned}\tag{3.27}$$

Finally, $H_r\left(T^2\right)=0$ for all $r\geq 3$.

3.4 The path chain complex on n-cube

Define the n-cube as follows:

$$n\text{- cube}=I^n=\underbrace{I\Box I\Box...\Box I}_{n},\tag{3.28}$$

where $I=\{0\to 1\}$ and $n\in\mathbb{N}$.Hence, each vertex a of the n-cube can be identified with a binary sequence $\left(a_1,...,a_n\right)$. For example, $\mathbf{0}=\left(0,...,0\right)$ and $\mathbf{1}=\left(1,...,1\right)$ are the corners of the n-cube.

Proposition 3.9. *We have for any* $r \geq 0$

$$|\Omega_r(I^n)| = 2^{n-r}\binom{n}{r}, \tag{3.29}$$

$$\beta_r\left(I^n\right) = \begin{cases} 1, & r = 0 \\ 0, & r > 0 \end{cases}. \tag{3.30}$$

A more detailed description of the spaces $\Omega_r(I^n)$ will be given below.

Proof. Let us prove (3.29) by induction in n. If $n = 1$ then

$$|\Omega_0\left(I\right)| = 2,\ |\Omega_1\left(I\right)| = 1,\ |\Omega_r\left(I\right)| = 0 \text{ for } r \geq 2,$$

which matches (3.29).

For induction step from n to $n+1$, we use that $I^{n+1} = I^n \Box I$ and obtain by (3.24) and the induction hypothesis that

$$\left|\Omega_r\left(I^{n+1}\right)\right| = \sum_{\{p,q\geq 0,\, p+q=r\}} |\Omega_p(I^n)|\ |\Omega_q(I)| = \sum_{\{p,q\geq 0,\, p+q=r\}} 2^{n-p}\binom{n}{p}2^{1-q}\binom{1}{q}.$$

Since here $q = 0$ or $q = 1$, it follows that

$$\begin{aligned} \left|\Omega_r\left(I^{n+1}\right)\right| &= 2^{n-r}\binom{n}{r}2^1\binom{1}{0} + 2^{n-(r-1)}\binom{n}{r-1}2^0\binom{1}{1} \\ &= 2^{n-r+1}\left(\binom{n}{r} + \binom{n}{r-1}\right) \\ &= 2^{(n+1)-r}\binom{n+1}{r}, \end{aligned}$$

which was to be proved.

The identity (3.30) is proved similarly by induction using (3.25). ■

For two vertices a, b of the n-cube, there is an arrow $a \to b$ if $b_k = a_k + 1$ for exactly one value of k and $b_k = a_k$ for all other values of k. Denote

$$|a| = a_1 + ... + a_n.$$

We write $a \preceq b$ if there is an allowed path from a to b, that is

$$a \preceq b \ \Leftrightarrow\ a_k \leq b_k \text{ for all } k = 1, \ldots, n.$$

For any pair $a \preceq b$ consider an induced subgraph $D_{a,b}$ of the n-cube as follows:

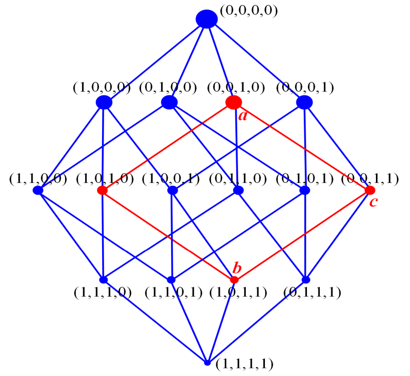
the vertices of $D_{a,b}$ are all vertices c of $I^{\Box n}$ such that

$$a \preceq c \preceq b$$

and an arrow $c_1 \to c_2$ exists in $D_{a,b}$ exactly when this arrow exists in $I^{\Box n}$.

Here is a 4-cube and its subgraph $D_{a,b}$:

(the arrows go from top to bottom).



The mapping $c \mapsto c - a$ provides an isomorphism of $D_{a,b}$ onto a p-cube with

$$p = |b| - |a| \,.$$

Assuming that $a \preceq b$, denote by $P_{a,b}$ the set of all elementary allowed paths going from a to b. All paths of $P_{a,b}$ lie in $D_{a,b}$, each path in $P_{a,b}$ has the length $p = |b| - |a|$, and the total number of the paths in $P_{a,b}$ is $p!$.

Lemma 3.10. *There is a function $\sigma : P_{a,b} \to \{0,1\}$ such that the following p-path*

$$\omega_{a,b} = \sum_{x \in P_{a,b}} (-1)^{\sigma(x)} e_x \tag{3.31}$$

is ∂-invariant.

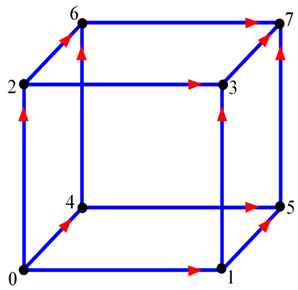
For example, in a 3-cube as shown here, we have

$$\omega_{0,1} = e_{01},$$

$$\omega_{0,3} = e_{013} - e_{023},$$

and

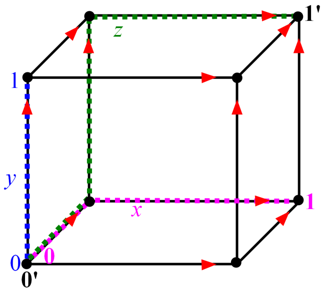
$$\omega_{0,7} = e_{0137} - e_{0237} - e_{0157} + e_{0457} + e_{0267} - e_{0467}$$



Proof. Without loss of generality, we can assume that $a = \mathbf{0}$, $b = \mathbf{1}$, and prove the claim by induction in $n = p$. The induction basis for $n = 1$ is obvious. For the induction step from n to $n+1$ we use Lemma 3.3 that says that the cross product of ∂-invariant paths is ∂-invariant. Denote by $\mathbf{0}' = (\mathbf{0}, 0)$ and $\mathbf{1}' = (\mathbf{1}, 1)$ the corners of the $(n+1)$-cube.

Taking the cross product of the n-path $\omega_{\mathbf{0},\mathbf{1}}$ on $I^{\square n}$ and the 1-path $y = e_{01}$ on I, and using (3.1), we obtain the following ∂-invariant $(n+1)$-path on $I^{\square(n+1)}$:

$$\omega_{\mathbf{0},\mathbf{1}} \times e_{01} = \sum_{x \in P_{\mathbf{0},\mathbf{1}}} (-1)^{\sigma(x)} e_x \times e_y$$
$$= \sum_{x \in P_{\mathbf{0},\mathbf{1}}} \sum_{z \in \Pi_{x,y}} (-1)^{\sigma(x)} (-1)^{L(z)} e_z,$$



A path $x \in P_{\mathbf{0},\mathbf{1}}$ and $z \in \Pi_{x,y}$

where z is any stair-like path on $(n+1)$-cube that projects onto x and y, respectively. Clearly, z runs over all paths $P_{\mathbf{0}',\mathbf{1}'}$. Setting

$$\sigma(z) = \sigma(x) + L(z) \bmod 2$$

and

$$\omega_{\mathbf{0}',\mathbf{1}'} = \omega_{\mathbf{0},\mathbf{1}} \times e_{01},$$

we obtain

$$\omega_{\mathbf{0}',\mathbf{1}'} = \sum_{z \in P_{\mathbf{0}',\mathbf{1}'}} (-1)^{\sigma(z)} e_z,$$

which concludes the proof. $\blacksquare$

Proposition 3.11. *For any $p \geq 0$, we have*

$$\Omega_p\left(I^n\right) = \left\langle \omega_{a,b} : a \preceq b \text{ and } |b| - |a| = p \right\rangle .$$

Moreover, $\{\omega_{a,b}\}$ is a basis of $\Omega_p\left(I^n\right)$.

Proof. The proof is again by induction in n. The induction basis for $n = 1$ is obvious. For the induction step from n to $n+1$ we use the Künneth formula (3.22). By this formula and by the induction hypothesis, we obtain that the basis in $\Omega_p\left(I^{n+1}\right)$ consists of the following p-paths:

$$\left\{\omega_{a,b} \times e_{01} : \omega_{a,b} \in \Omega_{p-1}\left(I^n\right)\right\} \cup \left\{\omega_{a,b} \times e_i : \omega_{a,b} \in \Omega_p\left(I^n\right), i = 0, 1\right\}$$

As above, the products $\omega_{a,b} \times e_{01}$ give us all the p-paths $\omega_{(a,0),(b,1)}$, while $\omega_{a,b} \times e_i$ give us all the p-paths $\omega_{(a,0),(b,0)}$ and $\omega_{(a,1),(b,1)}$. Clearly, we obtain in this way all p-paths $\omega_{a',b'}$ with $a', b' \in I^{n+1}$, which concludes the proof. ∎

4 Hodge Laplacian on digraphs

In this section we use the path chain complex without augmentation

$$\{0\} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \ldots$$

over the field $\mathbb{K} = \mathbb{R}$. Let us fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ in each of the spaces $\mathcal{R}_p$ so that we have an inner product also in all Ω_p. In all examples we use the natural inner product given by (2.12).

4.1 Definition of the Hodge Laplacian

For the operator $\partial : \Omega_p \to \Omega_{p-1}$, where $p \geq 0$, consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$. By the definition of an adjoint operator, we have

$$\langle \partial u, v \rangle = \langle u, \partial^* v \rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition. Define the *Hodge-Laplace operator* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u. \tag{4.1}$$

The pairs ∂^*, ∂ and ∂, ∂^* appearing in (4.1) are the following operators:

$$\Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p \quad \text{and} \quad \Omega_p \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_{p+1}.$$

Proposition 4.1. *The operator Δ_p is self-adjoint and non-negative definite.*

Proof. We have for all $u, v \in \Omega_p$

$$\langle \Delta_p u, v \rangle = \langle \partial^* \partial u + \partial \partial^* u, v \rangle = \langle \partial u, \partial v \rangle + \langle \partial^* u, \partial^* v \rangle = \langle u, \Delta_p v \rangle$$

so that Δ_p is self-adjoint, and

$$\langle \Delta_p u, u \rangle = \|\partial u\|^2 + \|\partial^* u\|^2 \geq 0, \tag{4.2}$$

so that $\Delta_p \geq 0$. ■

Hence, the spectrum of Δ_p is real, non-negative and consists of a finite sequence of eigenvalues.

4.2 Matrix of Δ_p

Let $\{\alpha_i\}$ be an orthonormal basis in Ω_p, $\{\beta_m\}$ be an orthonormal basis in Ω_{p-1} and $\{\gamma_n\}$ be an orthonormal basis in Ω_{p+1} :

$$\underset{\{\beta_m\}}{\Omega_{p-1}} \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \underset{\{\alpha_i\}}{\Omega_p} \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \underset{\{\gamma_n\}}{\Omega_{p+1}}.$$

The operator $\partial : \Omega_p \to \Omega_{p-1}$ has in the bases $\{\alpha_i\}$ and $\{\beta_m\}$ the matrix representation

$$B = (\langle \partial\alpha_i, \beta_m \rangle)_{m,i} \ , \tag{4.3}$$

where m is the row index and i is the column index.
Similarly, the operator $\partial^* : \Omega_p \to \Omega_{p+1}$ has the matrix representation

$$C = (\langle \partial^*\alpha_i, \gamma_n \rangle)_{n,i} = (\langle \alpha_i, \partial\gamma_n \rangle)_{n,i} \ , \tag{4.4}$$

where n is the row index and i is the column index. Since $\Delta_p = \partial^*\partial + (\partial^*)^* \partial^*$, we obtain the matrix of Δ_p in the basis $\{\alpha_i\}$:

$$A := \text{matrix of } \Delta_p = B^T B + C^T C. \tag{4.5}$$

More explicitly, the (i, j)-entry of the matrix A of Δ_p in the basis $\{\alpha_i\}$ is given by

$$A_{ij} = \sum_m \langle \partial\alpha_i, \beta_m \rangle \langle \partial\alpha_j, \beta_m \rangle + \sum_n \langle \alpha_i, \partial\gamma_n \rangle \langle \alpha_j, \partial\gamma_n \rangle \ , \tag{4.6}$$

where i is the row index and j is the column index.

Example. Recall that $\Omega_{-1} = \{0\}, \Omega_0 = \{e_i : i \in V\}$ and $\Omega_1 = \langle e_{kl} : k \to l \rangle$. Assuming that $\langle \cdot, \cdot \rangle$ is the natural inner product (2.12), we obtain by (4.6) that the matrix of Δ_0 is

$$\begin{aligned} A_{ij} &= \sum_{k\to l} \langle e_i, \partial e_{kl} \rangle \langle e_j, \partial e_{kl} \rangle \\ &= \sum_{k\to l} \langle e_i, e_l - e_k \rangle \langle e_j, e_l - e_k \rangle \\ &= \sum_{k\to l} (\delta_{il} - \delta_{ik}) (\delta_{jl} - \delta_{jk}) \\ &= \sum_{k\to i} \delta_{ij} + \sum_{i\to l} \delta_{ij} - \mathbf{1}_{\{i\to j\}} - \mathbf{1}_{\{j\to i\}} \\ &= \deg(i)\delta_{ij} - \mathbf{1}_{\{i\to j\}} - \mathbf{1}_{\{j\to i\}}. \end{aligned}$$

If G has no double arrow and $V = \{1, ..., n\}$ then

$$A = \operatorname{diag} (\deg (i))_{i=1}^n - \mathbf{1}_{\{i\sim j\}} \tag{4.7}$$

where $\mathbf{1}_{\{i\sim j\}}$ is the $n \times n$ adjacency matrix of G. Hence, Δ_0 is the usual unnormalized Laplacian (=Kirchhoff operator) on functions on V. Consequently, we have

$$\operatorname{trace} \Delta_0 = \sum_{i\in V} \deg (i) = 2 |E| \,. \tag{4.8}$$

4.3 Examples of computation of the matrix of Δ_1

In this section, we denote by V and E respectively the numbers of vertices and arrows of the digraph in question.

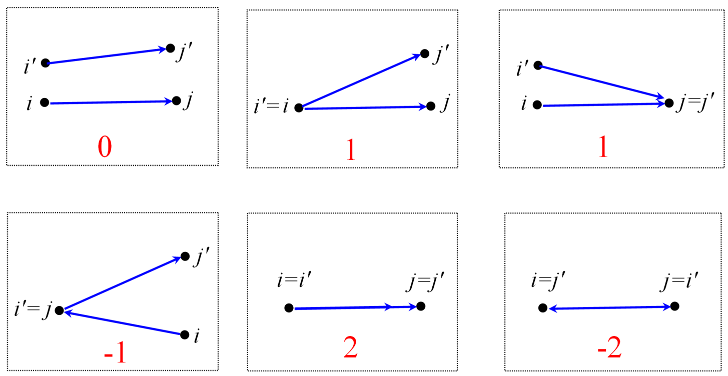
Let us compute Δ_1 for the natural inner product. We use the orthonormal bases $\{e_m\}$ in Ω_0 and $\{e_{ij} : i \to j\}$ in Ω_1. Let $\{\gamma_n\}$ be an orthonormal basis in Ω_2.
The matrix A of Δ_1 has dimensions $E \times E$ and, by (4.6), its entries are

$$A_{ij,i'j'} = \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle + \sum_n \langle e_{ij}, \partial \gamma_n \rangle \langle e_{i'j'}, \partial \gamma_n \rangle \tag{4.9}$$

for all arrows $i \to j$ and $i' \to j'$.
For the first sum in (4.9) we have

$$\begin{aligned} \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle &= \sum_m \langle e_j - e_i, e_m \rangle \langle e_{j'} - e_{i'}, e_m \rangle \\ &= \sum_m (\delta_{jm} - \delta_{im}) (\delta_{j'm} - \delta_{i'm}) \\ &= \delta_{jj'} - \delta_{ij'} - \delta_{ji'} + \delta_{ii'} =: [ij, i'j'] \,. \end{aligned}$$

The values of $[ij, i'j']$ are shown here:



Hence, in the case $p = 1$, we have

$$B^T B = ([ij, i'j']) \,. \tag{4.10}$$

In particular, diagonal entries of $B^T B$ are equal to 2.

Example. Consider an 1-torus $T = \{0 \to 1 \to 2 \to 0\}$. In this case we have $\Omega_1 = \langle e_{01}, e_{12}, e_{20} \rangle$ and

$$\begin{aligned} A = B^T B &= ([ij, i'j']) \\ &= \left(\begin{array}{cccc} & e_{01} & e_{12} & e_{20} \\ e_{01} & [01, 01] & [01, 12] & [01, 20] \\ e_{12} & [12, 01] & [12, 12] & [12, 20] \\ e_{20} & [20, 01] & [20, 12] & [20, 20] \end{array} \right) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} . \end{aligned}$$

The eigenvalues are

$$\operatorname{spec} \Delta_1(T) = (0, 3_2) \,,$$

where the subscript shows the multiplicity.

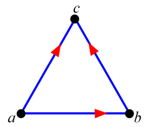
For a general digraph G with $\Omega_2 \neq \{0\}$, let us compute the entry $\langle e_{ij}, \partial \gamma_n \rangle$ of the matrix C assuming that $\gamma_n = \gamma$ is a triangle or square (note that although Ω_2 always has a basis of

triangles and squares, the squares in this basis do not have to be orthogonal).
If $\gamma = e_{abc}$ is a triangle then we have

$$\langle e_{ij}, \partial\gamma\rangle = \langle e_{ij}, e_{ab} + e_{bc} - e_{ac}\rangle = [ij, \gamma]\,, \tag{4.11}$$

where

$$[ij, \gamma] := \begin{cases} 1, & \text{if } ij \in \{ab, bc\} \\ -1 & \text{if } ij = ac \\ 0, & \text{otherwise.} \end{cases}$$

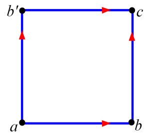


If $\gamma = \frac{e_{abc} - e_{ab'c}}{\sqrt{2}}$ is a (normalized) square then

$$\langle e_{ij}, \partial\gamma\rangle = \frac{1}{\sqrt{2}}\langle e_{ij},\, e_{ab} + e_{bc} - e_{ab'} - e_{b'c}\rangle = \frac{1}{\sqrt{2}}[ij, \gamma]\,, \tag{4.12}$$

where

$$[ij, \gamma] = \begin{cases} 1, & \text{if } ij \in \{ab\,, bc\} \\ -1 & \text{if } ij \in \{ab', b'c\} \\ 0, & \text{otherwise.} \end{cases}$$



Example. Let G be a triangle $\{0 \to 1 \to 2, 0 \to 2\}$. Then $\Omega_1 = \langle e_{01}, e_{12}, e_{02}\rangle$ and

$$B^T B = ([ij, i'j']) = \begin{array}{c} \\ e_{01} \\ e_{12} \\ e_{02} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} e_{01} & e_{12} & e_{02} \end{array} \\ \begin{pmatrix} [01, 01] & [01, 12] & [01, 20] \\ [12, 01] & [12, 12] & [12, 20] \\ [02, 01] & [02, 12] & [02, 02] \end{pmatrix} \end{array} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single triangle $\gamma = e_{012}$ so that

$$C = \begin{array}{c} \\ e_{012} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} e_{01} & e_{12} & e_{02} \end{array} \\ \begin{pmatrix} [01, \gamma] & [12, \gamma] & [02, \gamma] \end{pmatrix} \end{array} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix},$$

$$C^T C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

$$A = B^T B + C^T C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example. Let G be a square $\{0 \to 1 \to 3, 0 \to 2 \to 3\}$. Then $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}\rangle$ and

$$B^T B = ([ij, i'j']) = \begin{array}{c} \\ e_{01} \\ e_{02} \\ e_{13} \\ e_{23} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \end{array} \\ \begin{pmatrix} [01, 01] & [01, 02] & [01, 13] & [01, 23] \\ [02, 01] & [02, 02] & [02, 13] & [02, 23] \\ [12, 01] & [13, 02] & [13, 13] & [13, 23] \\ [23, 01] & [23, 02] & [23, 13] & [23, 23] \end{pmatrix} \end{array} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single square $\gamma = \frac{1}{\sqrt{2}}\left(e_{013} - e_{023}\right)$ so that

$$C = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \gamma \end{array} \!\!\begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \\ \end{array} \!\!\begin{pmatrix} [01,\gamma] & [02,\gamma] & [13,\gamma] & [23,\gamma] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix},$$

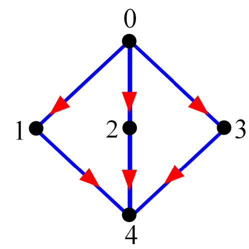
$$C^T C = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Hence,

$$A = B^T B + C^T C = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix},$$

and $\operatorname{spec} \Delta_1(\text{square}) = (2_3,\ 4)$.

Example. Consider the following digraph:



Here $V = 5,\ \ E = 6,\ \ |\Omega_2| = 2$ and

$$\Omega_2 = \langle e_{014} - e_{024}, e_{014} - e_{034} \rangle .$$

However, this basis is *not* orthogonal.
Orthogonalization gives an orthonormal basis for Ω_2:

$$\gamma_1 = \tfrac{1}{\sqrt{2}}\left(e_{014} - e_{024}\right),$$
$$\gamma_2 = \tfrac{1}{\sqrt{6}}\left(e_{014} + e_{024} - 2e_{034}\right).$$

Since

$$\partial\gamma_1 = \tfrac{1}{\sqrt{2}}\left(e_{01} + e_{14} - e_{02} - e_{24}\right),$$
$$\partial\gamma_2 = \tfrac{1}{\sqrt{6}}\left(e_{01} + e_{04} + e_{02} + e_{24} - 2e_{03} - 2e_{34}\right),$$

we obtain

$$C = (\langle e_{ij}, \partial\gamma_n\rangle) = \begin{array}{c} \\ \partial\gamma_1 \\ \partial\gamma_2 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} e_{01} & e_{14} & e_{02} & e_{24} & e_{03} & e_{34} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

and

$$C^T C = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

Now we compute $B^T B$:

$$B^T B = ([e_{ij}, e_{i'j'}]) = \begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix},$$

whence

$$A = B^T B + C^T C = \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} \end{pmatrix}.$$

Hence, $\operatorname{spec} \Delta_1(G) = (2_4, 3, 5)$.

Example. Let G be an $(n-1)$-simplex, that is, the vertices are $\{0, 1, ..., n-1\}$ and

$$i \to j \Leftrightarrow i < j.$$

Let us show that

$$A := \text{matrix of } \Delta_1 = \operatorname{diag}(n).$$

Let ij and $i'j'$ be two arrows. Then the $(ij, i'j')$-entry of A is

$$A_{ij,i'j'} = \left(B^T B\right)_{ij,i'j'} + \left(C^T C\right)_{ij,i'j'} = [ij, i'j'] + \sum_n [ij, \gamma_n] [i'j', \gamma_n], \tag{4.13}$$

where $\{\gamma_n\}$ is an orthonormal basis of Ω_2, which we may take to consist of all triangles in G. If $ij = i'j'$ then $[ij, i'j'] = 2$. Since the arrow ij belongs to $(n-2)$ triangles γ_n, we obtain

$$A_{ij,ij} = 2 + (n-2) = n,$$

that is, all the diagonal entries of Δ_1 are equal to n. It remains to show that if $ij \neq i'j'$ then

$$A_{ij,i'j'} = 0. \tag{4.14}$$

If ij and $i'j'$ have no common vertex then they cannot belong to the same triangle γ_n and, hence, all the terms in (4.13) vanish.

Suppose $i' = i$ and $j' \neq j$:

$$\begin{tikzcd} & \bullet^{j} \\ {}_{i'=i}\bullet \arrow[ur] \arrow[r] & \bullet^{j'} \end{tikzcd}$$

Then $[ij, i'j'] = 1$ while $[ij, \gamma_n] [i'j', \gamma_n]$ is nonzero only when γ_n is the triangle formed by i, j, j'. In this case the arrows ij and $i'j'$ have opposite orientations with respect to γ_n, whence $[ij, \gamma_n] [i'j', \gamma_n] = -1$ and (4.14) follows.

Suppose $j' = i$ and $i' \neq j$:

$$\begin{tikzcd} & \bullet^{j} \\ {}_{j'=i}\bullet \arrow[ur] & \bullet^{i'} \arrow[l] \end{tikzcd}$$

Then $[ij, i'j'] = -1$ while $[ij, \gamma_n][i'j', \gamma_n]$ is nonzero only when γ_n is the triangle $i'ij$. In this case the arrows ij and $i'j'$ have the same orientation with respect to γ_n, whence $[ij, \gamma_n][i'j', \gamma_n] = 1$ and again (4.14) follows.
The cases $j = i'$ and $j = j'$ are similar.

Problem. Describe all the digraphs for which Δ_1 has only one eigenvalue.

Problem. Devise a program for computing the matrix and spectrum of Δ_1 for large digraphs.

4.4 Harmonic paths

A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$.

Lemma 4.2. *A path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.*

Proof. Indeed, if $\partial u = 0$ and $\partial^* u = 0$ then by (4.1) we have $\Delta_p u = 0$. Conversely, if $\Delta_p u = 0$ then we obtain by (4.2) that

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \langle \Delta_p u, u \rangle = 0,$$

whence $\|\partial u\| = \|\partial^* u\| = 0$. ■
Denote by $\mathcal{H}_p$ the set of all harmonic paths in Ω_p, so that $\mathcal{H}_p$ is a subspace of Ω_p.

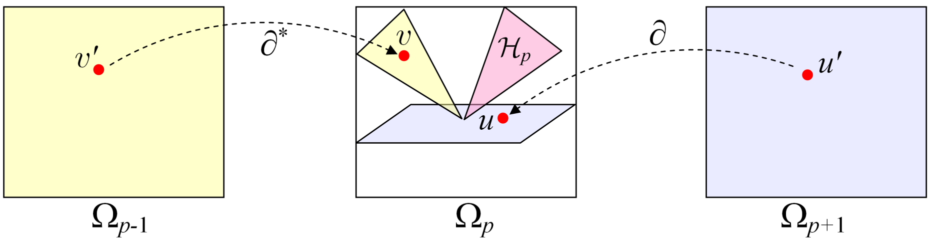
Theorem 4.3. (Hodge decomposition) *The space Ω_p is an orthogonal sum:*

$$\Omega_p = \partial \Omega_{p+1} \bigoplus \partial^* \Omega_{p-1} \bigoplus \mathcal{H}_p. \tag{4.15}$$

Proof. If $u \in \partial \Omega_{p+1}$ and $v \in \partial^* \Omega_{p-1}$ then $u = \partial u'$ and $v = \partial^* v'$, and we have

$$\langle u, v \rangle = \langle \partial u', \partial^* v' \rangle = \left\langle \partial^2 u', v' \right\rangle = 0,$$

so that the subspaces $\partial \Omega_{p+1}$ and $\partial^* \Omega_{p-1}$ are orthogonal.



Next, we have for any $w \in \Omega_p$

$$\begin{aligned} w \in (\partial^* \Omega_{p-1})^\perp &\Leftrightarrow \langle w, v \rangle = 0 \, \forall v \in \partial^* \Omega_{p-1} \\ &\Leftrightarrow \langle \partial w, v' \rangle = 0 \;\; \forall v' \in \Omega_{p-1} \\ &\Leftrightarrow \partial w = 0 \end{aligned}$$

and

$$w \in (\partial \Omega_{p+1})^\perp \Leftrightarrow \langle w, \partial u' \rangle = 0 \;\; \forall u' \in \Omega_{p+1}$$

$$
\begin{aligned}
&\Leftrightarrow \langle \partial^* w, u' \rangle = 0 \\
&\Leftrightarrow \partial^* w = 0 \\
&\Leftrightarrow w \in \mathcal{H}_p.
\end{aligned}
$$

Therefore, $w \in K := \left(\partial\Omega_{p+1} \bigoplus \partial^*\Omega_{p-1}\right)^{\perp}$ if and only if $\partial w = 0$ and $\partial^* w = 0$, that is, if $\Delta_p w = 0$, whence the identity $K = \mathcal{H}_p$ follows. ∎

Corollary 4.4. *There is a natural linear isomorphism*

$$H_p \cong \mathcal{H}_p. \tag{4.16}$$

In particular, $\dim \mathcal{H}_p = \beta_p$*; that is, the multiplicity of* 0 *as an eigenvalue of* Δ_p *is equal to the Betti number* β_p*.*

Proof. By the argument from the previous proof, a path $w \in \Omega_p$ belongs to $Z_p = \ker \partial|_{\Omega_{p+1}}$ if and only if $w \in \left(\partial^*\Omega_{p-1}\right)^{\perp}$. Since by (4.15)

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \bigoplus \partial^*\Omega_{p-1}$$

we obtain

$$Z_p = \left(\partial^*\Omega_{p-1}\right)^{\perp} = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \tag{4.17}$$

whence $\mathcal{H}_p \cong Z_p / \partial\Omega_{p+1} = H_p$. ∎

Remark. It follows from this argument that $\mathcal{H}_p$ is an orthogonal complement of B_p in Z_p and that any homology class $\omega \in H_p$ has a unique harmonic representative $u \in \mathcal{H}_p$. In addition, u minimizes the norm $\|\cdot\|$ among all representatives of ω.

5 Spectrum of the Hodge Laplacian

5.1 Trace of Δ_1

Recall that by (4.8)

$$\operatorname{trace} \Delta_0 = \sum_{i \in V} \deg(i) = 2E,$$

where E denotes the number of arrows. Here is a similar result for the trace of Δ_1.

Theorem 5.1. *Let* T *be the number of triangles in* G*,* S *be the number of linearly independent squares in* G*, and* D *be the number of double arrows* $a \rightleftarrows b$*. Then*

$$\operatorname{trace} \Delta_1 = 2E + 3T + 2S + 4D. \tag{5.1}$$

Recall that by a triangle we mean an allowed 2-path e_{abc} such that $a \to c$, and by a square we mean an allowed 2-path $e_{abc} - e_{ab'c}$ such that $a \neq c$ and $a \not\to c$.

Proof. Let $\{\alpha_i\}$ be the sequence of all arrows e_{ij} in G that forms an orthonormal basis in Ω_1, $\{\beta_m\}$ be the sequence of all vertices e_i that forms an orthonormal basis in Ω_0. Let $\{\gamma_n\}$ be an orthonormal basis in Ω_2. As above, consider the matrices B and C with entries

$$B_{mi} = \langle \partial\alpha_i, \beta_m \rangle \quad \text{and} \quad C_{ni} = \langle \partial\gamma_n, \alpha_i \rangle,$$

so that the matrix A of Δ_1 in the basis $\{\alpha_i\}$ is

$$A = B^T B + C^T C.$$

Consequently,

$$\operatorname{trace} \Delta_1 = \operatorname{trace} B^T B + \operatorname{trace} C^T C.$$

As we have seen above (see (4.10)), in the case of Δ_1, all the diagonal entries of $B^T B$ are equal to 2 so that

$$\operatorname{trace} B^T B = 2E.$$

Let us prove that

$$\operatorname{trace} C^T C = \sum_n \|\partial \gamma_n\|^2 . \tag{5.2}$$

We clearly have

$$\left(C^T C\right)_{ij} = \sum_n C_{ni} C_{nj} = \sum_n \langle \partial \gamma_n, \alpha_i \rangle \langle \partial \gamma_n, \alpha_j \rangle ,$$

whence it follows that

$$\operatorname{trace} C^T C = \sum_i \sum_n \langle \partial \gamma_n, \alpha_i \rangle^2 = \sum_n \sum_i \langle \partial \gamma_n, \alpha_i \rangle^2 = \sum_n \|\partial \gamma_n\|^2 ,$$

whence (5.2) follows.

For what follows, let $\{\gamma_n\}$ be an orthogonal basis. Then (5.2) transforms to

$$\operatorname{trace} C^T C = \sum_n \frac{\|\partial \gamma_n\|^2}{\|\gamma_n\|^2}. \tag{5.3}$$

As we know, Ω_2 has a basis $\{\gamma_n\}$ that consists of triangles, squares and double arrows. The only non-orthogonal pairs in this basis are pairs of squares containing the same elementary 2-path, like $e_{abc} - e_{ab'c}$ and $e_{abc} - e_{ab''c}$. Assume first that the entire basis $\{\gamma_n\}$ is orthogonal (which is equivalent to absence of multisquares).

A double arrow $a \rightleftarrows b$ gives two elements of the basis $\{\gamma_n\}$: e_{aba} and e_{bab}. If $\gamma_n = e_{aba}$ then

$$\|\gamma_n\|^2 = 1, \quad \partial \gamma_n = e_{ba} + e_{ab}, \quad \|\partial \gamma_n\|^2 = 2$$

and

$$\frac{\|\partial \gamma_n\|^2}{\|\gamma_n\|^2} = 2.$$

The same is true for $\gamma_n = e_{bab}$ so that each double arrow contributes 4 to the sum

$$\sum_n \frac{\|\partial \gamma_n\|^2}{\|\gamma_n\|^2}. \tag{5.4}$$

If γ_n is a triangle e_{abc} then

$$\|\gamma_n\|^2 = 1, \quad \partial \gamma_n = e_{bc} - e_{ac} + e_{ab}, \quad \|\partial \gamma_n\|^2 = 3,$$

whence

$$\frac{\|\partial \gamma_n\|^2}{\|\gamma_n\|^2} = 3,$$

so that each triangle contributes 3 to the sum (5.4).

If γ_n is a square $e_{abc} - e_{ab'c}$ then

$$\|\gamma_n\|^2 = 2, \quad \partial \gamma_n = e_{ab} + e_{bc} - e_{ab'} - e_{b'c}, \quad \|\partial \gamma_n\|^2 = 4,$$

so that

$$\frac{\|\partial \gamma_n\|^2}{\|\gamma_n\|^2} = 2,$$

so that each square contributes 2 to the sum (5.4). Hence, we obtain that the sum (5.4) is equal to $3T + 2S + 4D$, which proves (5.1) in this case.

In the general case G may contain multisquares. Assume that G contains the following m-square

$$a, \ \{b_k\}_{k=0}^m, \ c$$

which gives rise to m linearly independent squares:

$$e_{ab_0c} - e_{ab_1c}, \ e_{abc} - e_{ab_2c}, \ \ldots, \ e_{abc} - e_{ab_mc}\,. \tag{5.5}$$

The sequence (5.5) is not orthogonal, and its orthogonalization gives the following sequence:

$$\begin{aligned}
\omega_1 &= e_{ab_0c} - e_{ab_1c} \\
\omega_2 &= e_{ab_0c} + e_{ab_1c} - 2e_{ab_2c} \\
&\ldots \\
\omega_k &= e_{ab_0c} + \ldots + e_{ab_{k-1}c} - ke_{ab_kc} \\
&\ldots \\
\omega_m &= e_{ab_0c} + \ldots + e_{ab_{m-1}c} - me_{ab_mc}
\end{aligned}$$

We have

$$\partial \omega_k = (e_{ab_0} + e_{b_0c}) + \ldots + \left(e_{ab_{k-1}} + e_{b_{k-1}c}\right) - k\left(e_{ab_k} + e_{b_kc}\right)$$
$$\|\partial \omega_k\|^2 = 2k + 2k^2, \quad \|\omega_k\|^2 = k + k^2,$$

whence

$$\frac{\|\partial \omega_k\|^2}{\|\omega_k\|^2} = 2.$$

Hence, each ω_k contributes 2 to the sum (5.4), which completes the proof. ■

Since the sum of all eigenvalues is $\operatorname{trace} \Delta_1$ and the eigenvalue 0 has the multiplicity β_1, we obtain that the average of the positive eigenvalues is

$$\lambda_{average} = \frac{\operatorname{trace} \Delta_1}{E - \beta_1}.$$

5.2 An upper bound on $\lambda_{\max}(\Delta_1)$

Denote by $\lambda_{\max}(A)$ the maximal eigenvalue of a symmetric operator A. Let $\langle\cdot,\cdot\rangle$ is the natural inner product in all spaces $\mathcal{R}_p$.

Proposition 5.2. *We have*

$$\lambda_{\max}(\Delta_0) \le 2\max_{i\in V}\deg(i).$$

Proof. Set $D = \max_{i\in V}\deg(i)$. By the variational principle, it suffices to prove that for all $u \in \Omega_0$

$$\frac{\langle\Delta_0 u, u\rangle}{\|u\|^2} \le 2D.$$

Since $\partial u = 0$, we have by (4.2)

$$\langle\Delta_0 u, u\rangle = \|\partial^* u\|^2.$$

Since for any $i \to j$

$$\langle\partial^* u, e_{ij}\rangle = \langle u, \partial e_{ij}\rangle = \langle u, e_j - e_i\rangle = u^j - u^i,$$

it follows that

$$\|\partial^* u\|^2 = \sum_{i\to j}(u^j-u^i)^2 \le 2\sum_{i\to j}(u^j)^2 + 2\sum_{i\to j}(u^i)^2 = 2\sum_i \deg(i)(u^i)^2 \le 2D\|u\|^2, \quad (5.6)$$

whence the claim follows. ∎

Note that the bottom eigenvalue of Δ_0 is always 0 because if all $u^k = 1$ then by (5.6) $\partial^* u = 0$ and, hence, $\Delta_0 u = \partial\partial^* u = 0$. If G a complete bipartite graph $K_{D,D}$, then G is D-regular and $2D$ is the top eigenvalue of Δ_0.

For any arrow $i \to j$ in G denote by $\deg_\Delta(ij)$ the number of triangles containing the arrow $i \to j$, and by $\deg_\square(ij)$ the number of squares containing $i \to j$.

Theorem 5.3. *Assume that G has no 2-squares or double arrows. Then*

$$\lambda_{\max}(\Delta_1) \le 2\max_i \deg(i) + 3\max_{i\to j}\deg_\Delta(ij) + 2\max_{i\to j}\deg_\square(ij). \quad (5.7)$$

Proof. The hypothesis means that Ω_2 has a basis that consists of all squares and triangles; in this case this basis is orthogonal.

Recall that

$$\lambda_{\max}(\Delta_1) = \sup_{u\in\Omega_1\setminus\{0\}}\left(\frac{\|\partial u\|^2}{\|u\|^2} + \frac{\|\partial^* u\|^2}{\|u\|^2}\right).$$

Since the operators $\partial : \Omega_1 \to \Omega_0$ and $\partial^* : \Omega_0 \to \Omega_1$ are dual, they have the same norm. The norm of the latter was estimated in the proof of Proposition 5.2 (cf. (5.6)), whence we obtain the same estimate for the norm of the former, that is, for any non-zero $u \in \Omega_1$,

$$\frac{\|\partial u\|^2}{\|u\|^2} \le 2\max_{i\in V}\deg(i).$$

Let us prove that

$$\frac{\|\partial^* u\|^2}{\|u\|^2} \leq 3 \max_{i\to j} \deg_\Delta (ij) + 2 \max_{i\to j} \deg_\Box (ij) . \tag{5.8}$$

Let $u = \sum_{i\to j} u^{ij} e_{ij}$ and, hence,

$$\|u\|^2 = \sum_{i\to j} (u^{ij})^2$$

Let $\{\gamma_n\}$ be the orthogonal basis in Ω_2 that consists of all squares and triangles. We have

$$\|\partial^* u\|^2 = \sum_n \frac{\langle \partial^* u, \gamma_n \rangle^2}{\|\gamma_n\|^2} = \sum_n \frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2}.$$

If γ_n is a triangle e_{abc} then $\|\gamma_n\| = 1$,

$$\langle u, \partial\gamma_n \rangle = \langle u, e_{bc} - e_{ac} + e_{ab} \rangle = u^{bc} - u^{ac} + u^{ab},$$

$$\langle u, \partial\gamma_n \rangle^2 \leq 3 \left((u^{bc})^2 + (u^{ac})^2 + (u^{ab})^2 \right) .$$

Summing up over all triangles γ_n and using that any arrow $i \to j$ occurs in $\deg_\Delta (ij)$ triangles, we obtain

$$\sum_{n:\gamma_n \text{ is a triangle}} \frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2} \leq 3 \sum_{i\to j} (u^{ij})^2 \deg_\Delta (ij) \leq 3 \|u\|^2 \max_{i\to j} \deg_\Delta (ij) . \tag{5.9}$$

Let now γ_n be a square $e_{abc} - e_{ab'c}$ (such that $a \not\to c$). Then $\|\gamma_n\|^2 = 2$,

$$\langle u, \partial\gamma_n \rangle = \langle u, e_{ab} + e_{bc} - e_{ab'} + e_{b'c} \rangle = u^{ab} + u^{bc} - u^{ab'} - u^{b'c},$$

$$\langle u, \partial\gamma_n \rangle^2 \leq 4 \left((u^{ab})^2 + (u^{bc})^2 + (u^{ab'})^2 + (u^{b'c})^2 \right) .$$

Summing up over all squares γ_n and using that any arrow $i \to j$ occurs in $\deg_\Box (ij)$ squares, we obtain

$$\begin{aligned} \sum_{n:\gamma_n \text{ is a square}} \frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2} &\leq 2 \sum_{i\to j} (u^{ij})^2 \deg_\Box (ij) \\ &\leq 2 \|u\|^2 \max_{i\to j} \deg_\Box (ij) . \end{aligned} \tag{5.10}$$

Adding up (5.9) and (5.10), we obtain (5.8). ■

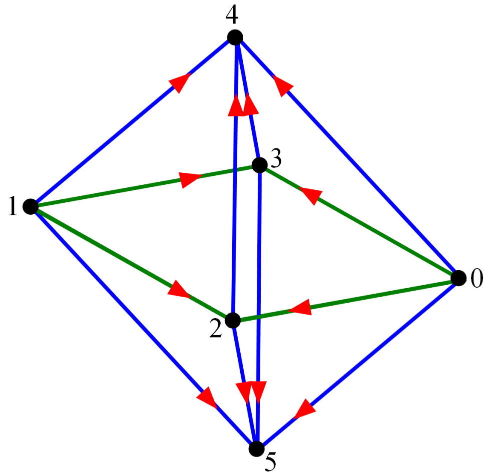
Problem. How sharp is the upper bound on $\lambda_{\max} (\Delta_1)$ in (5.7)? Is it attained on some digraphs? Extend (5.7) to the general case when a basis of triangles and squares requires orthogonalization.

Problem. Find reasonable upper bounds for $\lambda_{\max}(\Delta_p)$ in terms of geometric and combinatorial quantities of G. The question amounts to obtaining an upper bound for the Rayleigh quotient for non-zero $u \in \Omega_p$:

$$\frac{\|\partial u\|^2 + \|\partial^* u\|^2}{\|u\|^2} \leq ?$$

5.3 Examples of computations of $\operatorname{spec}\Delta_1$

Example. Consider an octahedron based on a diamond:



For this digraph $V = 6, \quad E = 12, \quad |\Omega_2| = 8.$
The space Ω_2 is generated by 8 triangles:

$$\Omega_2 = \langle e_{024}\,, e_{025}\,, e_{034}\,, e_{035}\,, e_{124}\,, e_{125}\,, e_{134}\,, e_{135} \rangle.$$

We have $T = 8, \; S = 0$, which implies

$$\operatorname{trace}\Delta_1 = 2E + 3T = 48.$$

Since $\beta_1 = 0$, it follows that

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{48}{12} = 4.$$

Using (4.10), we obtain

$$B^T B = \begin{array}{cc} & \begin{array}{cccccccccccc} e_{02} & e_{03} & e_{12} & e_{13} & e_{04} & e_{14} & e_{24} & e_{34} & e_{05} & e_{15} & e_{25} & e_{35} \end{array} \\ \begin{array}{c} e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{04} \\ e_{14} \\ e_{24} \\ e_{34} \\ e_{05} \\ e_{15} \\ e_{25} \\ e_{35} \end{array} & \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 2 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

for example,

$$\left(B^T B\right)_{12} = [02, 03] = 1 \quad \text{and} \quad \left(B^T B\right)_{17} = [02, 24] = -1.$$

Using (4.4) and (4.11), we obtain that

$$C^TC = \begin{array}{cc} & \begin{array}{cccccccccccc} e_{02} & e_{03} & e_{12} & e_{13} & e_{04} & e_{14} & e_{24} & e_{34} & e_{05} & e_{15} & e_{25} & e_{35} \end{array} \\ \begin{array}{c} e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{04} \\ e_{14} \\ e_{24} \\ e_{34} \\ e_{05} \\ e_{15} \\ e_{25} \\ e_{35} \end{array} & \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

for example,

$$\left(C^TC\right)_{12} = \sum_n \langle e_{02}, \partial\gamma_n\rangle \langle e_{03}, \partial\gamma_n\rangle = \sum_n [02, \gamma_n]\,[03, \gamma_n] = 0$$

and

$$\left(C^TC\right)_{17} = \sum_n [02, \gamma_n]\,[24, \gamma_n] = 1.$$

Since the matrix of Δ_1 is $A = B^TB + C^TC$, we obtain

$$A = \begin{array}{cc} & \begin{array}{cccccccccccc} e_{02} & e_{03} & e_{12} & e_{13} & e_{04} & e_{14} & e_{24} & e_{34} & e_{05} & e_{15} & e_{25} & e_{35} \end{array} \\ \begin{array}{c} e_{02} \\ e_{03} \\ e_{12} \\ e_{13} \\ e_{04} \\ e_{14} \\ e_{24} \\ e_{34} \\ e_{05} \\ e_{15} \\ e_{25} \\ e_{35} \end{array} & \begin{pmatrix} 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{pmatrix} \end{array}$$

The eigenvalues of Δ_1 are

$$\operatorname{spec}\Delta_1 = (2_3,\ 4_6,\ 6_3)\,,$$

where the subscript denotes the multiplicity.

The eigenvalue 2 has the eigenvectors

$$e_{02} - e_{03} - e_{12} + e_{13}$$
$$e_{04} - e_{14} - e_{05} + e_{15}$$
$$e_{24} - e_{34} - e_{25} + e_{35}$$

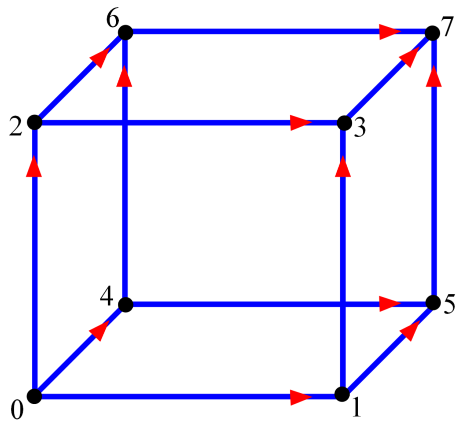
the eigenvalue 6 has the eigenvectors

$$\begin{array}{l} e_{02}+e_{03}+e_{12}+e_{13} \\ e_{04}+e_{14}+e_{05}+e_{15} \\ e_{24}+e_{34}+e_{25}+e_{35} \end{array}$$

the eigenvalue 4 has the eigenvectors

$$e_{03}-e_{12},\ e_{02}-e_{13},\ e_{14}-e_{05},\ e_{04}-e_{15},\ e_{34}-e_{25},\ e_{24}-e_{35}$$

Example. Consider a 3-cube:



We have $V=8,\quad E=12,\quad |\Omega_2|=6$,
$H_p=\{0\}$ for $p\geq 1$.

Space Ω_2 is generated by 6 squares, so that

$$S=6 \text{ and } T=0.$$

Hence, we obtain by (5.1)

$$\operatorname{trace}\Delta_1 = 2E+2S = 2\cdot 12+2\cdot 6 = 36.$$

Since $\beta_1=0$, we obtain

$$\lambda_{average}=\frac{\operatorname{trace}\Delta_1}{E-\beta_1}=3.$$

The eigenvalues of Δ_1 on a 3-cube are

$$\operatorname{spec}\Delta_1(\text{3-cube}) = (2_6,\ 3_2,\ 4_3,\ 6).$$

Example. Consider the n-cube, that is,

$$I^n=\underbrace{I\Box I\Box ...\Box I}_{n \text{ times}}$$

where $I=\{0\rightarrow 1\}$. Using the Künneth formula for product, it is possible to prove by induction that, for any $p\geq 0$

$$|\Omega_p(I^n)| = 2^{n-p}\binom{n}{p},$$

and that $\Omega_p(I^n)$ is generated by all sub-cubes of I^n of dimension p. Also by induction we obtain that

$$\beta_p(I^n)=\left\{\begin{array}{ll} 1, & p=0 \\ 0, & p>0 \end{array}\right. .$$

(see Subsection 3.4). Hence, we have

$$V = 2^n, \quad E = n2^{n-1}, \quad S = |\Omega_2| = 2^{n-3}n\,(n-1)$$

and $T = D = 0$. It follows that

$$\operatorname{trace}\Delta_1(I^n) = 2E + 2S = 2^{n-2}n\,(n+3)$$

and

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1(I^n)}{E - \beta_1} = \frac{2^{n-2}n\,(n+3)}{n2^{n-1}} = \frac{n+3}{2}.$$

For example, for the 4-cube we obtain

$$\operatorname{trace}\Delta_1(I^4) = 2^2 \cdot 4 \cdot 7 = 112.$$

The eigenvalues of Δ_1 on the 4-cube are

$$\operatorname{spec}\Delta_1(I^4) = (2_{10},\ 3_8,\ 4_9,\ 6_4,\ 8).$$

For the 5-cube we obtain

$$\operatorname{trace}\Delta_1(I^5) = 2^3 \cdot 5 \cdot 8 = 320.$$

The eigenvalues of Δ_1 on the 5-cube are

$$\operatorname{spec}\Delta_1(I^5) = (2_{15},\ 3_{20},\ 4_{25},\ 5_4,\ 6_{10},\ 8_5,\ 10).$$

Note that $\Delta_1(I^5)$ acts on the space $\Omega_1(I^5)$ that has dimension $= 5 \cdot 2^{5-1} = 80$.

Problem. Determine the full spectrum of $\Delta_1(I^n)$. In particular, prove that

$$\lambda_{\max} = 2n \ \text{ and } \ \lambda_{\min} = 2_{\frac{n(n+1)}{2}}.$$

Prove that $\operatorname{spec}\Delta_1(I^n)$ consists of all even integers from 2 to $2n$ and of all odd integers from 3 to n. How to compute their multiplicities?

The difficulty here is that the method of separation of variables does not work for Δ_1 on Cartesian products.

Example. Consider the 2-torus $T^2 = T\Box T$ where $T = \{0 \to 1 \to 2 \to 0\}$.

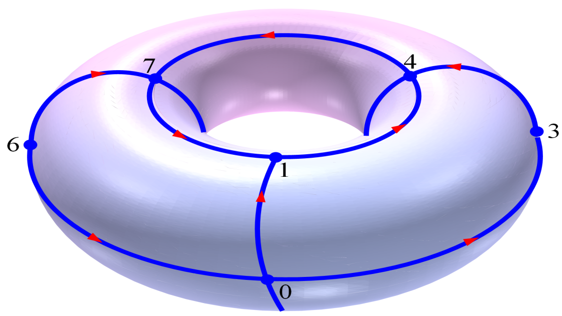
Here $V = 9, \ \ E = 18, \ \ |\Omega_2| = 9, |H_1| = 2.$

Space Ω_2 is generated by 9 squares, whence

$$\operatorname{trace}\Delta_1(T^2) = 2 \cdot 18 + 2 \cdot 9 = 54.$$

The eigenvalues of Δ_1 on the 2-torus are

$$\operatorname{spec}\ \Delta_1(T^2) = (0_2,\ (\tfrac{3}{2})_4,\ 3_8,\ 6_4).$$



For the 3-torus $T^3 = T\Box T\Box T$ we have

$$E = 81, \ \ S = |\Omega_2| = 81, \ \ |H_1| = 3,$$

whence

$$\operatorname{trace}\Delta_1(T^3) = 2\cdot 81 + 2\cdot 81 = 324.$$

The eigenvalues of Δ_1 on the 3-torus are

$$\operatorname{spec}\Delta_1(T^3) = (0_3,\ (\tfrac{3}{2})_{12},\ 3_{30},\ (\tfrac{9}{2})_{16},\ 6_{12},\ 9_8).$$

For the n-torus

$$T^n = \underbrace{T\square T\square ...\square T}_{n \text{ times}}$$

we have

$$E = n3^n,\quad S = |\Omega_2| = \frac{n\,(n-1)}{2}3^n,\quad |H_1| = n,$$

whence

$$\operatorname{trace}\Delta_1(T^n) = 2E + 2S = n\,(n+1)\,3^n$$

and

$$\lambda_{average} = \frac{n\,(n+1)\,3^n}{n3^n - n} = (n+1)\,\frac{3^n}{3^n - 1}.$$

Problem. Compute the full spectrum of $\Delta_1\,(T^n)$. In particular, prove that

$$\lambda_{\max} = (3n)_{2^n}\,.$$

In fact, $\lambda_{\min} = 0_n$, which is a consequence of $\beta_1(T^n) = n$.

5.4 Eigenvalues of Δ_1 on trapezohedron

A *trapezohedron* T_m of order $m \geq 2$ is a configuration of $2m+2$ distinct vertices

$$a,\ b,\ i_0,\ldots,i_{m-1},\ j_0,\ldots,j_{m-1}$$

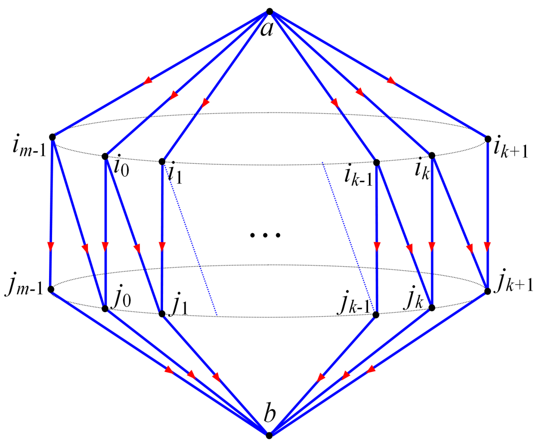
with $4m$ arrows:

$$a\to i_k,\quad j_k\to b$$

and

$$i_k\to j_k,\quad i_k\to j_{k+1},$$

for all $k = 0,\ldots,m-1$, where k is understood $\operatorname{mod} m$.



The trapezohedron gives rise to the following ∂-invariant 3-path:

$$\tau_m = \sum_{k=0}^{m-1}\left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b}\right). \tag{5.11}$$

Indeed, τ_m is clearly allowed, and its boundary is also allowed because

$$\partial\tau_m = \sum_{k=0}^{m-1}\partial\left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b}\right)$$

$$= \sum_{k=0}^{m-1} \left(e_{i_k j_k b} - e_{i_k j_{k+1} b} \right) - \sum_{k=0}^{m-1} \left(e_{a i_k j_k} - e_{a i_k j_{k+1}} \right) \tag{5.12}$$

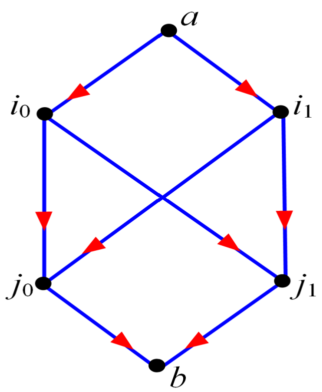
$$- \sum_{k=0}^{m-1} \left(e_{a j_k b} - e_{a j_{k+1} b} \right) + \sum_{k=0}^{m-1} \left(e_{a i_k b} - e_{a i_k b} \right), \tag{5.13}$$

where the both sums in (5.12) are allowed, while the both sums in (5.13) vanish.

A trapezohedron T_2 is shown here:

In this case we have

$$\tau_2 = e_{a i_0 j_0 b} - e_{a i_0 j_1 b} + e_{a i_1 j_1 b} - e_{a i_1 j_0 b}.$$



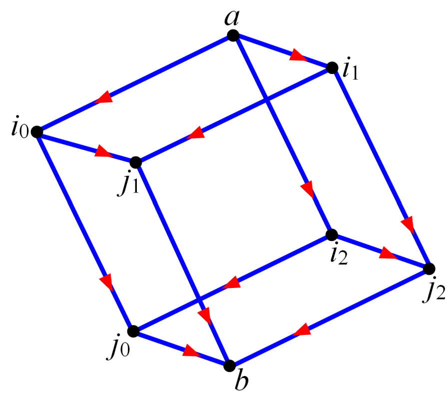
Any trapezohedron of order $m \geq 3$ can be realized as a convex polyhedron in $\mathbb{R}^3$ with flat faces.

Here is T_3 that clearly coincides with 3-cube:

In this case we have

$$\tau_3 = e_{a i_0 j_0 b} - e_{a i_0 j_1 b} + e_{a i_1 j_1 b} - e_{a i_1 j_2 b}$$
$$+ e_{a i_2 j_2 b} - e_{a i_2 j_0 b}.$$

The path τ_3 coincides (up to a sign) with the aforementioned 3-path determined by 3-cube.



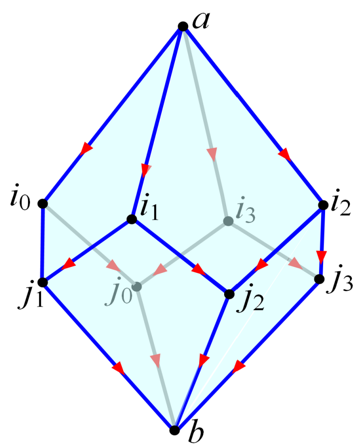
A trapezohedron T_4 is shown here:

As a polyhedron in $\mathbb{R}^3$, it is called

tetragonal trapezohedron.

In this case we have

$$\tau_4 = e_{a i_0 j_0 b} - e_{a i_0 j_1 b} + e_{a i_1 j_1 b} - e_{a i_1 j_2 b}$$
$$+ e_{a i_2 j_2 b} - e_{a i_2 j_3 b} + e_{a i_3 j_3 b} - e_{a i_3 j_0 b}.$$



The significance of trapezohedra in our theory is determined by the following theorem.

Theorem 5.4. *Assume that the digraph G contains neither 2-squares nor double arrows. Then $\Omega_3(G)$ is generated by trapezohedral paths τ_m with $m \geq 2$ and their digraph images.*

The hypothesis here means, in particular, that G does not contain linearly dependent squares.

Consequently, $\Omega_2(G)$ has a basis that consists of all squares and triangles.

Conjecture. The claim of this theorem is true for an arbitrary digraph G.

In the rest of this section we consider T_m as a digraph and are concerned with the eigenvalues of $\Delta_1(T_m)$. By Theorem 1.5, $\Omega_2(T_m)$ is spanned by all $2m$ squares

$$e_{ai_{k-1}j_k} - e_{ai_kj_k} \text{ and } e_{i_kj_kb} - e_{i_kj_{k+1}b}, \quad k = 0, 1, ..., m-1.$$

One can show that $\Omega_3(T_m) = \langle \tau_m \rangle$. Obviously, $\Omega_p(T_m) = \{0\}$ for all $p \geq 4$. One can also show that $H_p(T_m) = \{0\}$ for all $p \geq 1$.
Since the number of arrows in T_m is $E = 4m$, the number of squares is $S = 2m$, and there are no triangles or double arrows, we obtain by Theorem 5.1, that

$$\operatorname{trace} \Delta_1(T_m) = 2 \cdot 4m + 2 \cdot 2m = 12m,$$

and, hence,

$$\lambda_{average} = \frac{\operatorname{trace} \Delta_1}{E - \beta_1} = \frac{12m}{4m} = 3.$$

Example. The following results are based on numerical computation of the matrix of $\Delta_1(T_m)$.
Case $m = 2$:

$$\operatorname{spec} \Delta_1(T_2) = (2,\ 3_5,\ \tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{17}).$$

Case $m = 3$:

$$\operatorname{spec} \Delta_1(T_3) = (2_6,\ 3_2,\ 4_3,\ 6).$$

Case $m = 4$:

$$\operatorname{spec} \Delta_1(T_4) = \{2,\ 3_4,\ 5,\ \tfrac{9}{2} \pm \tfrac{1}{2}\sqrt{17},\ (2 \pm \tfrac{1}{2}\sqrt{2})_2,\ (3 \pm \sqrt{2})_2\},$$

and the characteristic polynomial of $\Delta_1(T_4)$ is

$$(z-2)\,(z-3)^4\,(z-5)\,(z^2 - 9z + 16)(z^2 - 4z + \tfrac{7}{2})^2(z^2 - 6z + 7)^2.$$

Case $m = 5$:

$$\operatorname{spec} \Delta_1(T_5) = \{2,\ (\tfrac{5}{2})_4,\ 6,\ 5 \pm \sqrt{5},\ (\tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{5})_2,\ (\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{5})_2,\ (2 \pm \tfrac{1}{2}\sqrt{5})_2\},$$

and the characteristic polynomial of $\Delta_1(T_5)$ is

$$(z-2)\,(z-\tfrac{5}{2})^4\,(z-6)\,(z^2 - 10z + 20)(z^2 - 7z + 11)^2(z^2 - 5z + 5)^2(z^2 - 4z + \tfrac{11}{4})^2.$$

Case $m = 6$:

$$\operatorname{spec} \Delta_1(T_6) = (2_5,\ 3_7,\ 4_2,\ 7,\ 8,\ (\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{3})_2,\ (3 \pm \sqrt{3})_2),$$

and the characteristic polynomial of $\Delta_1(T_6)$ is

$$(z-2)^5\,(z-3)^7\,(z-4)^2\,(z-7)\,(z-8)\,(z^2 - 3z + \tfrac{3}{2})^2(z^2 - 6z + 6)^2.$$

Case $m = 7$: the characteristic polynomial of $\Delta_1(T_7)$ is

$$(z-2)\,(z-8)\,(z^2-12z+28)(z^3-6z^2+\tfrac{41}{4}z-\tfrac{29}{8})^2(z^3-10z^2+31z-29)^2$$
$$\times\,(z^3-7z^2+\tfrac{63}{4}z-\tfrac{91}{8})^2(z^3-8z^2+19z-13)^2.$$

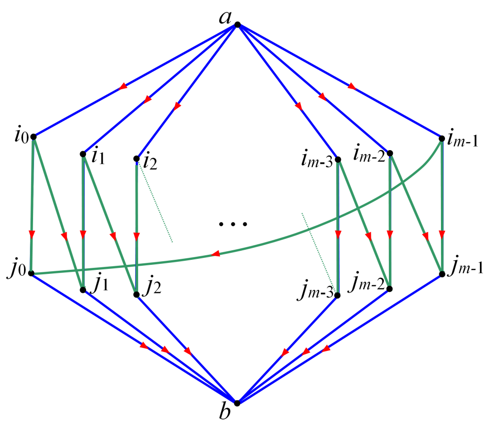
It has eigenvalues 2 and 8, and all other eigenvalues are irrational.

Problem. Determine the full spectrum of Δ_1 on the trapezohedron T_m for any m. In particular, what are $\lambda_{\min}$ and $\lambda_{\max}$?

Here is a partial answer.

Proposition 5.5. *For any $m \geq 2$, the operator $\Delta_1(T_m)$ has eigenvalues $\lambda = 2$ and $\lambda = m+1$.*

Proof. The vertices of T_m will be denoted as here:



Consider the following 1-paths on T_m:

$$v = e_{i_0j_1} + e_{i_1j_2} + \ldots + e_{i_{m-1}j_0}$$
$$-\left(e_{i_0j_0} + e_{i_1j_1} + \ldots + e_{i_{m-1}j_{m-1}}\right)$$
$$= \sum_{k=0}^{m-1}\left(e_{i_{k-1}j_k} - e_{i_kj_k}\right),$$

where the index k is regarded $\operatorname{mod} m$, and

$$u = e_{ai_0} + e_{ai_1} + \ldots + e_{ai_{m-1}}$$
$$-\left(e_{j_0b} + e_{j_1b} + \ldots + e_{j_{m-1}b}\right)$$
$$= \sum_{k=0}^{m-1}\left(e_{ai_k} - e_{j_kb}\right).$$

The 1-paths u and v are obviously allowed and, hence, ∂-invariant. We will prove that

$$\Delta_1 v = 2v \quad \text{and} \quad \Delta_1 u = (m+1)\,u,$$

which will settle the claim. We have clearly

$$\partial v = \sum_{k=0}^{m-1}\left(e_{j_k} - e_{i_{k-1}} - e_{j_k} + e_{i_k}\right) = 0,$$

and, hence, $\partial^*\partial v = 0$.

In order to compute $\partial^* v \in \Omega_2$ we use the following orthogonal basis in Ω_2 that consists of all $2m$ squares in T_m:

$$\varphi_k = e_{ai_{k-1}j_k} - e_{ai_kj_k} \quad \text{and} \quad \psi_k = e_{i_kj_kb} - e_{i_kj_{k+1}b},$$

where $k = 0, \ldots, m-1$. We have for any k

$$\langle \partial^* v, \varphi_k\rangle = \langle v, \partial\varphi_k\rangle = \left\langle v, e_{i_{k-1}j_k} + e_{ai_{k-1}} - e_{i_kj_k} - e_{ai_k}\right\rangle = 2,$$
$$\langle \partial^* v, \psi_k\rangle = \langle v, \partial\psi_k\rangle = \left\langle v, e_{j_kb} + e_{i_kj_k} - e_{j_{k+1}b} - e_{i_kj_{k+1}}\right\rangle = -2,$$

which together with $\|\varphi_k\|^2 = \|\psi_k\|^2 = 2$ implies that

$$\partial^* v = \sum_{k=0}^{m-1}\left(\varphi_k - \psi_k\right).$$

Hence, we obtain

$$\begin{aligned}
\Delta_1 v = \partial\partial^* v &= \sum_{k=0}^{m-1} (\partial\varphi_k - \partial\psi_k) \\
&= \sum_{k=0}^{m-1} (e_{i_{k-1}j_k} + e_{ai_{k-1}} - e_{i_kj_k} - e_{ai_k}) \\
&\quad - \sum_{k=0}^{m-1} (e_{j_kb} + e_{i_kj_k} - e_{j_{k+1}b} - e_{i_kj_{k+1}}) \\
&= 2\sum_{k=0}^{m-1} (e_{i_{k-1}j_k} - e_{i_kj_k}) = 2v.
\end{aligned}$$

Next, let us compute $\partial^* u$. We have for any k,

$$\begin{aligned}
\langle \partial^* u, \varphi_k \rangle &= \langle u, \partial\varphi_k \rangle = \left\langle u, e_{i_{k-1}j_k} + e_{ai_{k-1}} - e_{i_kj_k} - e_{ai_k} \right\rangle = 0, \\
\langle \partial^* u, \psi_k \rangle &= \langle u, \partial\psi_k \rangle = \left\langle u, e_{j_kb} + e_{i_kj_k} - e_{j_{k+1}b} - e_{i_kj_{k+1}} \right\rangle = 0,
\end{aligned}$$

whence $\partial^* u = 0$ and, hence, $\partial\partial^* u = 0$. It remains to compute $\partial^*\partial u$. We have

$$\partial u = \sum_{k=0}^{m-1} (e_{i_k} - e_a - e_b + e_{j_k}) = \sum_{k=0}^{m-1} (e_{i_k} + e_{j_k}) - m\,(e_a + e_b)\,.$$

For any 0-path e_i and any 1-path $e_{\alpha\beta}$ we have

$$\langle \partial^* e_i, e_{\alpha\beta} \rangle = \langle e_i, \partial e_{\alpha\beta} \rangle = \langle e_i, e_\beta - e_\alpha \rangle = \delta_{i\beta} - \delta_{i\alpha}$$

whence

$$\partial^* e_i = \sum_{\alpha\to\beta} (\delta_{i\beta} - \delta_{i\alpha})\, e_{\alpha\beta} = \sum_{\alpha\to i} e_{\alpha i} - \sum_{i\to\beta} e_{i\beta}.$$

It follows that

$$\begin{aligned}
\partial^* e_{i_k} &= e_{ai_k} - e_{i_kj_k} - e_{i_kj_{k+1}}, \\
\partial^* e_{j_k} &= e_{i_{k-1}j_k} + e_{i_kj_k} - e_{j_kb}, \\
\partial^* e_a = -\sum_{k=0}^{m-1} e_{ai_k}, &\quad \partial^* e_b = \sum_{k=0}^{m-1} e_{j_kb},
\end{aligned}$$

whence

$$\begin{aligned}
\Delta_1 u = \partial^*\partial u &= \sum_{k=0}^{m-1} (e_{ai_k} - e_{i_kj_k} - e_{i_kj_{k+1}} + e_{i_{k-1}j_k} + e_{i_kj_k} - e_{j_kb}) \\
&\quad + m \sum_{k=0}^{m-1} (e_{ai_k} - e_{j_kb}) \\
&= (m+1) \sum_{k=0}^{m-1} (e_{ai_k} - e_{j_kb}) = (m+1)\, u,
\end{aligned}$$

which finishes the proof. ■

5.5 Spectrum of Δ_p on join

In this section we use again the augmented chain complex (2.3):

$$\mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{5.14}$$

We also use the natural inner product $\langle \cdot, \cdot \rangle$ of paths that was defined by (2.12).
Denote by $\widetilde{\Delta}_p$ the Hodge Laplacian associated with this complex. Of course, $\widetilde{\Delta}_p$ coincides with Δ_p for $p \geq 1$ but is different for $p = -1$ and $p = 0$.
For example, we have for the chain complex (5.14)

$$\langle \partial^* e, e_i \rangle = \langle e, \partial e_i \rangle = \langle e, e \rangle = 1$$

so that

$$\partial^* e = \sum_{j \in V} e_j =: \sigma,$$

whence

$$\widetilde{\Delta}_{-1} e = \partial \partial^* e = \partial \sigma = |V| \, e.$$

In particular,

$$\operatorname{spec} \widetilde{\Delta}_{-1} = \{|V|\} \, . \tag{5.15}$$

In the case $p = 0$ we have

$$\widetilde{\Delta}_0 e_i = \partial^* \partial e_i + \partial \partial^* e_i = \partial^* e + \Delta_0 e_i = \Delta_0 e_i + \sigma,$$

that is,

$$\langle \widetilde{\Delta}_0 e_i, e_j \rangle = \langle \Delta_0 e_i, e_j \rangle + 1. \tag{5.16}$$

Therefore, the matrix of $\widetilde{\Delta}_0$ is obtained from the matrix of Δ_0 by adding 1 to *each* entry. For any $u \in \Omega_0$ we have

$$\widetilde{\Delta}_0 u = \Delta_0 u + \left(\sum_{k \in V} u^k \right) \sigma.$$

The advantage of using the chain complex (5.14) lies in the following statements.

Proposition 5.6. *Let X, Y be two digraphs. Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ with $p.q \geq -1$ we have*

$$\widetilde{\Delta}_r (u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v, \tag{5.17}$$

where $r = p + q + 1$.

For the proof we need the following lemma.

Lemma 5.7. *For all $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with $p, q \geq -1$ we have*

$$\partial^* (u * v) = (\partial^* u) * v + (-1)^{p+1} u * (\partial^* v) \, . \tag{5.18}$$

Proof. Set $Z = X * Y$. By definition of ∂^*, we have, for any $w \in \Omega_{p+q+2}(Z)$,

$$\langle \partial^* (u * v), w \rangle = \langle u * v, \partial w \rangle.$$

By Theorem 2.7, any $w \in \Omega_{p+q+2}(Z)$ admits a representation

$$w = \sum_j \varphi_j * \psi_j$$

where the sum is finite and

$$\varphi_j \in \Omega_{p_j}(X) \text{ and } \psi_j \in \Omega_{q_j}(Y)$$

with $p_j + q_j + 1 = p + q + 2$. Using (2.5) and (2.13), we obtain

$$\begin{aligned}
\langle \partial^* (u * v), w \rangle &= \langle u * v, \sum \partial (\varphi_j * \psi_j) \rangle \\
&= \langle u * v, \sum (\partial \varphi_j * \psi_j + (-1)^{p_j+1} \varphi_j * \partial \psi_j) \rangle \\
&= \sum \langle u * v, \partial \varphi_j * \psi_j \rangle + (-1)^{p_j+1} \langle u * v, \varphi_j * \partial \psi_j \rangle \\
&= \sum \langle u, \partial \varphi_j \rangle \langle v, \psi_j \rangle + (-1)^{p_j+1} \langle u, \varphi_j \rangle \langle v, \partial \psi_j \rangle \\
&= \sum \langle \partial^* u, \varphi_j \rangle \langle v, \psi_j \rangle + (-1)^{p_j+1} \langle u, \varphi_j \rangle \langle \partial^* v, \psi_j \rangle.
\end{aligned}$$

Note that if $p_j \neq p$ then $\langle u, \varphi_j \rangle = 0$. Hence, we can replace p_j everywhere by p and obtain

$$\begin{aligned}
\langle \partial^* (u * v), w \rangle &= \sum \langle \partial^* u, \varphi_j \rangle \langle v, \psi_j \rangle + (-1)^{p+1} \langle u, \varphi_j \rangle \langle \partial^* v, \psi_j \rangle \\
&= \sum \langle \partial^* u * v, \varphi_j * \psi_j \rangle + \langle (-1)^{p+1} u * \partial^* v, \varphi_j * \psi_j \rangle \\
&= \langle \partial^* u * v + (-1)^{p+1} u * \partial^* v, \sum \varphi_j * \psi_j \rangle \\
&= \langle \partial^* u * v + (-1)^{p+1} u * \partial^* v, w \rangle.
\end{aligned}$$

Since this identity is true for any $w \in \Omega_{p+q+2}(Z)$, we obtain (5.18). ■

Proof of Proposition 5.6. By (5.18) we have

$$\begin{aligned}
\partial \partial^* (u * v) &= \partial \left(\partial^* u * v + (-1)^{p+1} u * \partial^* v \right) \\
&= \partial (\partial^* u * v) + (-1)^{p+1} \partial (u * \partial^* v) \\
&= \partial \partial^* u * v + (-1)^{p+2} \partial^* u * \partial v \\
&+ (-1)^{p+1} \left(\partial u * \partial^* v + (-1)^{p+1} u * \partial \partial^* v \right) \\
&= \partial \partial^* u * v + (-1)^p \partial^* u * \partial v + (-1)^{p+1} \partial u * \partial^* v + u * \partial \partial^* v
\end{aligned}$$

and by (2.5)

$$\begin{aligned}
\partial^* \partial (u * v) &= \partial^* \left(\partial u * v + (-1)^{p+1} u * \partial v \right) \\
&= \partial^* (\partial u * v) + (-1)^{p+1} \partial^* (u * \partial v) \\
&= \partial^* \partial u * v + (-1)^p \partial u * \partial^* v \\
&+ (-1)^{p+1} \left(\partial^* u * \partial v + (-1)^{p+1} u * \partial^* \partial v \right) \\
&= \partial^* \partial u * v + (-1)^p \partial u * \partial^* v + (-1)^{p+1} \partial^* u * \partial v + u * \partial^* \partial v.
\end{aligned}$$

Adding up the two identities, we see that the terms $\partial^* u * \partial v$ and $\partial u * \partial^* v$ cancel out, and we obtain (5.17). ■

Theorem 5.8. *Let X, Y be two digraphs. We have for any $r \geq 0$*

$$\operatorname{spec} \widetilde{\Delta}_r (X * Y) = \bigsqcup_{\{p,q \geq -1 : p+q=r-1\}} \left(\operatorname{spec} \widetilde{\Delta}_p (X) + \operatorname{spec} \widetilde{\Delta}_q (Y) \right). \tag{5.19}$$

Here we denote by $\operatorname{spec} A$ a sequence of all the eigenvalues of the operator A counted with multiplicities. The sum of two such sequences consists of all pairwise sums of the elements of the sequences, and the disjoint union of sequences means the union of all sequences, summing up the multiplicities. In particular, if one of the sequences is empty then its sum with another sequence is also empty.

Proof of Theorem 5.8. Observe that if $u \in \Omega_p (X)$ and $v \in \Omega_q (Y)$ are eigenvectors such that

$$\widetilde{\Delta}_p u = \lambda u \text{ and } \widetilde{\Delta}_q v = \mu v,$$

then we have by (5.17) for $r = p + q + 1$:

$$\widetilde{\Delta}_r (u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v = (\lambda + \mu) (u * v),$$

that is, $u * v$ is an eigenvector of $\widetilde{\Delta}_r$ on $X * Y$ with the eigenvalue $\lambda + \mu$.
In each $\Omega_p (X)$ there is a basis that consists of eigenvectors of $\widetilde{\Delta}_p$; denote by $\{u_k\}$ the union of all such bases of $\Omega_p (X)$ across all $p \geq -1$, with the corresponding eigenvalues $\{\lambda_k\}$. Let $\{v_l\}$ be a similar sequence on Y with the eigenvalues $\{\mu_l\}$. By Theorem 2.3, we have, for any $r \geq -1$,

$$\Omega_r (X * Y) \cong \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} (\Omega_p (X) \otimes \Omega_q (Y)),$$

that is, $\Omega_r (X * Y)$ has a basis

$$\{u_k * v_l : |u_k| + |v_l| = r - 1\}.$$

The elements of this basis are the eigenvectors of $\widetilde{\Delta}_r$ on $X * Y$ with eigenvalues $\lambda_k + \mu_l$, whence (5.19) follows. ∎
In particular, for $r = 0$ we have

$$\begin{aligned}
\operatorname{spec} \widetilde{\Delta}_0 (X * Y) &= \left(\operatorname{spec} \widetilde{\Delta}_{-1} (X) + \operatorname{spec} \widetilde{\Delta}_0 (Y) \right) \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0 (X) + \operatorname{spec} \widetilde{\Delta}_{-1} (Y) \right) \\
&= \left(\{|X|\} + \operatorname{spec} \widetilde{\Delta}_0 (Y) \right) \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0 (X) + \{|Y|\} \right)
\end{aligned} \tag{5.20}$$

and for $r = 1$

$$\begin{aligned}
\operatorname{spec} \widetilde{\Delta}_1 (X * Y) = & \left(\operatorname{spec} \widetilde{\Delta}_{-1} (X) + \operatorname{spec} \widetilde{\Delta}_1 (Y) \right) \\
& \sqcup \left(\operatorname{spec} \widetilde{\Delta}_1 (X) + \operatorname{spec} \widetilde{\Delta}_{-1} (Y) \right) \\
& \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0 (X) + \operatorname{spec} \widetilde{\Delta}_0 (Y) \right).
\end{aligned}$$

Since $\widetilde{\Delta}_1 = \Delta_1$, we conclude that

$$\operatorname{spec} \Delta_1 (X * Y) = (\{|X|\} + \operatorname{spec} \Delta_1 (Y))$$

$$
\begin{aligned}
&\sqcup \left(\operatorname{spec}\Delta_1 (X) + \{|Y|\}\right) \\
&\sqcup \left(\operatorname{spec}\widetilde{\Delta}_0 (X) + \operatorname{spec}\widetilde{\Delta}_0 (Y)\right). \qquad (5.21)
\end{aligned}
$$

Example. Let K_n be the complete digraph of n vertices, that is, the directed $(n-1)$-simplex. Let us show that, for any $-1 \leq p \leq n-1$,

$$
\operatorname{spec}\widetilde{\Delta}_p (K_n) = \{n\}, \tag{5.22}
$$

where we neglect the multiplicity that is equal to $|\Omega_p(K_n)| = \binom{n}{p+1}$. If $p = -1$ then we have by (5.15)

$$
\operatorname{spec}\widetilde{\Delta}_{-1} (K_n) = \{|K_n|\} = \{n\}.
$$

Hence, assume in the sequel that $p \geq 0$ and prove (5.22) by induction in n.

Induction basis for $n = 1$. Then necessarily $p = 0$. By (5.16), the matrix of $\widetilde{\Delta}_0(K_1)$ consists of a single entry 1, so that

$$
\operatorname{spec}\widetilde{\Delta}_0(K_1) = \{1\}.
$$

Induction step from n to $n+1$. Since $K_{n+1} = K_n * K_1$ we obtain by (5.19) that, for any $0 \leq r \leq n$,

$$
\begin{aligned}
\operatorname{spec}\widetilde{\Delta}_r (K_{n+1}) &= \bigsqcup_{\{p,q \geq -1: p+q=r-1\}} \left(\operatorname{spec}\widetilde{\Delta}_p (K_n) + \operatorname{spec}\widetilde{\Delta}_q (K_1)\right) \\
&= \left(\operatorname{spec}\widetilde{\Delta}_r (K_n) + \operatorname{spec}\widetilde{\Delta}_{-1} (K_1)\right) \sqcup \left(\operatorname{spec}\widetilde{\Delta}_{r-1} (K_n) + \operatorname{spec}\widetilde{\Delta}_0 (K_1)\right) \\
&= \{n+1\} \sqcup \{n+1\} = \{n+1\},
\end{aligned}
$$

where we did not count the multiplicity.

5.6 Spectrum of Δ_p on digraphs D_m^n

We compute here $\operatorname{spec}\Delta_p (D_m^n)$ where D_m^n is the n-th join power of the digraph D_m that consists of m disjoint vertices (without edges).

Lemma 5.9. *We have*

$$
|\Omega_{r-1} (D_m^n)| = \tbinom{n}{r} m^r. \tag{5.23}
$$

Proof. Induction in n. The induction basis for $n = 1$ is obvious as the only non-zero values of $|\Omega_{r-1}(D_m)|$ occur for the following values of r:

- for $r = 0$, $|\Omega_{-1}(D_m)| = 1$;
- for $r = 1$, $|\Omega_0(D_m)| = m$.

For the induction step we use $D_m^{n+1} = D_m^n * D_m$ and Theorem 2.3 that yields

$$
\left|\Omega_{r-1} \left(D_m^{n+1}\right)\right| = \sum_{\{p,q \geq 0: p+q=r\}} |\Omega_{p-1}(D_m^n)| \, |\Omega_{q-1}(D_m)|.
$$

Using the induction hypothesis and the induction basis, we obtain, for $q = 0$ and $q = 1$, that

$$
\left|\Omega_{r-1} \left(D_m^{n+1}\right)\right| = \binom{n}{r} m^r \cdot 1 + \binom{n}{r-1} m^{r-1} \cdot m = \binom{n+1}{r} m^r,
$$

which finishes the proof. ∎

It follows from Lemma 5.9 that the Hodge Laplacian Δ_{r-1} on D_m^n is non-trivial only if $n \geq r$.

Theorem 5.10. *We have, for all $n, m \geq 1$ and $r \geq 2$,*

$$\operatorname{spec} \Delta_{r-1}(D^n_m) = \left\{ ((n-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k} \right\}_{k=0}^{r}. \tag{5.24}$$

More explicitly, (5.24) can be stated as follows: if $n < r$ then

$$\operatorname{spec} \Delta_{r-1}(D^n_m) = \emptyset,$$

while for $n \geq r$ the spectrum of $\Delta_{r-1}(D^n_m)$ consists of the following $r+1$ eigenvalues

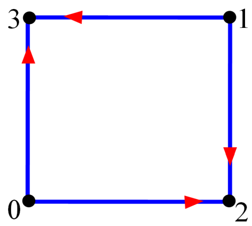
$$(n-r)m,\ (n-r+1)m,\ (n-r+2)m, ..., (n-1)m,\ nm, \tag{5.25}$$

having the following multiplicities:

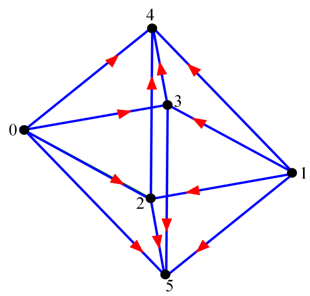
$$\binom{n}{r}(m-1)^r,\ \ r\binom{n}{r}(m-1)^{r-1},\ \binom{r}{2}\binom{n}{r}(m-1)^{r-2}, ..., r\binom{n}{r}(m-1),\ \binom{n}{r}. \tag{5.26}$$

Example. Let $m = 1$. Clearly, D^n_1 coincides with a complete digraph K_n (that is, an $(n-1)$-simplex digraph). In this case all the multiplicities in (5.26) are 0 except for the last one $\binom{n}{r}$. Hence, $\operatorname{spec} \Delta_{r-1}(K_n)$ consists of a single eigenvalue n with multiplicity $\binom{n}{r}$.

Example. Let $m = 2$. Then $D^n_2 = S^{n-1}$ can be regarded as a digraph sphere of dimension $n-1$. For example, S^1 is a diamond and S^2 is an octahedron:



$S^1 = D^2_2$ is a diamond



$S^2 = D^3_2$ is an octahedron

In this case (5.24) becomes

$$\operatorname{spec} \Delta_{r-1}(S^{n-1}) = \left\{ (2(n-k))_{\binom{r}{k}\binom{n}{r}} \right\}_{k=0}^{r}.$$

For example, for $r = 2$ we have

$$\operatorname{spec} \Delta_1(S^{n-1}) = \left\{ (2(n-2))_{\binom{n}{2}}, (2(n-1))_{2\binom{n}{2}}, (2n)_{\binom{n}{2}} \right\},$$

for $r = 3$

$$\operatorname{spec} \Delta_2(S^{n-1}) = \left\{ (2(n-3))_{\binom{n}{3}}, (2(n-2))_{3\binom{n}{3}}, (2(n-1))_{3\binom{n}{3}}, (2n)_{\binom{n}{3}} \right\},$$

and for $r = 4$

$$\operatorname{spec} \Delta_3(S^{n-1}) \ \ = \ \ \left\{ (2(n-4))_{\binom{n}{4}},\ (2(n-3))_{4\binom{n}{4}},\ (2(n-2))_{6\binom{n}{4}}, \right.$$

$$(2(n-1))_{4\binom{n}{4}},\ (2n)_{\binom{n}{4}}\Big\}\,.$$

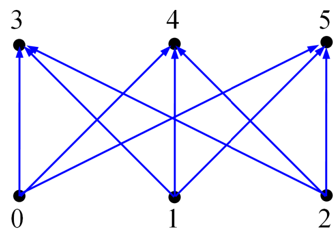
In particular,

$$\operatorname{spec}\Delta_1(S^1)=\{0,\ 2_2,\ 4\}\,,$$

$$\operatorname{spec}\Delta_1(S^2)=\{2_3,4_6,\ 6_3\},\quad \operatorname{spec}\Delta_2(S^2)=\{0,\ 2_3,\ 4_3,\ 6\}\,,$$

$$\begin{aligned}\operatorname{spec}\Delta_1(S^3) &= \{4_6,\ 6_{12},\ 8_6\},\ \operatorname{spec}\Delta_2(S^3)=\{2_4,\ 4_{12},\ 6_{12},\ 8_4\}\,,\\ \operatorname{spec}\Delta_3(S^3) &= \{0,\ 2_4,\ 4_6,\ 6_4,\ 8\}\,.\end{aligned}$$

Example. Let $m=3$ and $n=2$. Then D_3^2 coincides with the complete bipartite digraph $K_{3,3}$.



Digraph $K_{3,3}$

In this case (5.24) yields for $r=2$ that

$$\operatorname{spec}\Delta_1(K_{3,3})=\Big\{(3(2-k))_{\binom{2}{k}\binom{2}{2}2^k}\Big\}_{k=0}^{2}=\{0_4,\ 3_4,\ 6\}\,.$$

Proof of Theorem 5.10. Since $r-1\geq 1$, we have $\Delta_{r-1}=\widetilde{\Delta}_{r-1}$ so that (5.24) is equivalent to

$$\operatorname{spec}\widetilde{\Delta}_{r-1}(D_m^n)=\Big\{((n-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k}\Big\}_{k=0}^{r}. \tag{5.27}$$

We will prove that (5.27) holds for all $r\geq 0$ and $n,m\geq 1$. Consider first the case $r=0$. Then the left hand side of (5.27) is

$$\operatorname{spec}\widetilde{\Delta}_{-1}(D_m^n)=\{|D_m^n|\}=\{nm\}\,, \tag{5.28}$$

while the right hand side consists of a single value nm with the multiplicity 1, so that (5.27) holds for $r=0$.

Let us prove (5.27) for all $r\geq 1$ (and $m\geq 1$) by induction in n.

Induction basis for $n=1$. If $r=1$ then the right hand side of (5.27) is

$$\{0_{m-1},m\}\,.$$

On the other hand, we have by (4.7) that $\Delta_0(D_m)=0$, and by (5.16) the matrix of $\widetilde{\Delta}_0(D_m)$ is an $m\times m$ matrix with all entries $=1$.It follows that

$$\operatorname{spec}\widetilde{\Delta}_0(D_m)=\{0_{m-1},m\}\,, \tag{5.29}$$

which proves (5.27) in this case. If $r \geq 2$ then

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m) = \emptyset,$$

and the right hand side of (5.27) is also empty as all the multiplicities vanish. Hence, we have verified (5.27) for $n = 1$.

For the induction step from n to $n+1$, let us use that

$$D_m^{n+1} = D_m^n * D_m,$$

and

$$|D_m| = m, \quad |D_m^n| = nm.$$

Let us apply (5.19) and rewrite it in the form

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^{n+1}) = \bigsqcup_{\{p,q\geq 0: p+q=r\}} \left(\operatorname{spec} \widetilde{\Delta}_{p-1}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)\right).$$

Spectrum $\operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)$ is empty if $q \geq 2$. Hence, the values of q should be restricted to $q = 0$ and $q = 1$. Applying (5.28) and (5.29) to compute $\operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)$ for $q = 0$ and $q = 1$ as well as the induction hypothesis (5.27) to compute $\operatorname{spec} \widetilde{\Delta}_{p-1}\left(D_m^n\right)$ for $p = r$ and $p = r-1$, we obtain

$$\begin{aligned}
&\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^{(n+1)}) \\
&= \left(\operatorname{spec} \widetilde{\Delta}_{r-1}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{-1}\left(D_m\right)\right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_{r-2}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{0}\left(D_m\right)\right) \\
&= \left\{((n-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k} + m\right\}_{k=0}^{r} && (5.30)\\
&\quad \sqcup \left\{((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l} + \{0_{m-1}, m\}\right\}_{l=0}^{r-1}. && (5.31)
\end{aligned}$$

The sequence in (5.30) is equal to

$$\left\{((n+1-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k}\right\}_{k=0}^{r}, \tag{5.32}$$

and the sequence in (5.31) is equal to

$$\begin{aligned}
&\left\{((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^{l+1}}, ((n-l+1)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l}\right\}_{l=0}^{r-1} \\
&= \left\{((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^{l+1}}\right\}_{l=0}^{r-1} \\
&\quad \sqcup \left\{((n+1-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l}\right\}_{l=0}^{r-1} \\
&= \left\{((n+1-k)m)_{\binom{r-1}{k-1}\binom{n}{r-1}(m-1)^k}\right\}_{k=1}^{r} \\
&\quad \sqcup \left\{((n+1-k)m)_{\binom{r-1}{k}\binom{n}{r-1}(m-1)^k}\right\}_{k=0}^{r-1} \\
&= \left\{((n+1-k)m)_{\binom{r-1}{k-1}\binom{n}{r-1}(m-1)^k}\right\}_{k=0}^{r}
\end{aligned}$$

$$\sqcup \left\{ ((n+1-k)m)_{\binom{r-1}{k}\binom{n}{r-1}(m-1)^k} \right\}_{k=0}^{r}. \tag{5.33}$$

Combining (5.32) and (5.33), we conclude that $\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^{(n+1)})$ consists of the eigenvalues

$$\lambda_k = (n+1-k)m, \quad k = 0, ..., r,$$

where the multiplicity of λ_k is

$$\begin{aligned}
&\left[\binom{r}{k}\binom{n}{r} + \binom{r-1}{k-1}\binom{n}{r-1} + \binom{r-1}{k}\binom{n}{r-1}\right](m-1)^k \\
&= \left[\binom{r}{k}\binom{n}{r} + \binom{r}{k}\binom{n}{r-1}\right](m-1)^k \\
&= \binom{r}{k}\binom{n+1}{r}(m-1)^k,
\end{aligned}$$

which proves the induction step. ■

6 Hodge Laplacian on Cartesian products

6.1 Weighted Hodge Laplacian

Let us fix a sequence of positive numbers $a = \{a_p\}_{p=0}^{\infty}$ and define a weighted inner product in Λ_p by

$$\left\langle e_{i_0 \dots i_p}, e_{j_0 \dots j_p} \right\rangle_a = \frac{1}{a_p} \left\langle e_{i_0 \dots i_p}, e_{j_0 \dots j_p} \right\rangle = \frac{1}{a_p} \delta_{i_0 \dots i_p}^{j_0 \dots j_p}.$$

In particular, $\left\| e_{i_0 \dots i_p} \right\|_a^2 = \frac{1}{a_p}$. Denote by $\Delta_p^{(a)}$ the corresponding Hodge Laplacian.

Lemma 6.1. *For any $u \in \Omega_p$ we have*

$$\Delta_p^{(a)} u = \frac{a_p}{a_{p-1}} \partial^* \partial u + \frac{a_{p+1}}{a_p} \partial \partial^* u,$$

where ∂^* refers to the adjoint operator with respect to the natural inner product.

Proof. We have for $v \in \Omega_{p-1}$ and $\omega \in \Omega_p$

$$\langle \partial_a^* v, \omega \rangle_a = \langle v, \partial \omega \rangle_a = \frac{1}{a_{p-1}} \langle v, \partial \omega \rangle = \frac{1}{a_{p-1}} \langle \partial^* v, \omega \rangle = \frac{a_p}{a_{p-1}} \langle \partial^* v, \omega \rangle_a$$

whence

$$\partial_a^* v = \frac{a_p}{a_{p-1}} \partial^* v \ \text{ for } v \in \Omega_{p-1}.$$

It follows that, for $u \in \Omega_p$,

$$\Delta_p^{(a)} u = \partial_a^* \partial u + \partial_a \partial^* u = \frac{a_p}{a_{p-1}} \partial^* \partial u + \frac{a_{p+1}}{a_p} \partial \partial^* u$$

because $\partial u \in \Omega_{p-1}$ and $u \in \Omega_p$. ■

The significance of the weighted Hodge Laplacian is determined by the following statement.

Proposition 6.2. *Let*

$$a_p = p!. \tag{6.1}$$

Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ and $r = p + q$,

$$\Delta_r^{(a)}(u \times v) = \Delta_p^{(a)} u \times v + u \times \Delta_q^{(a)} v.$$

Proof. The proof is based on the product rule (3.2) and on the identity

$$\langle u \times v, \varphi \times \psi \rangle_a = \langle u, \varphi \rangle_a \langle v, \psi \rangle_a .$$

Indeed, using (3.9) we obtain

$$\begin{aligned}
\langle u \times v, \varphi \times \psi \rangle_a &= \frac{1}{(p+q)!} \langle u \times v, \varphi \times \psi \rangle \\
&= \frac{1}{(p+q)!} \binom{p+q}{p} \langle u, \varphi \rangle \langle v, \psi \rangle \\
&= \frac{1}{p!} \frac{1}{q!} \langle u, \varphi \rangle \langle v, \psi \rangle \\
&= \langle u, \varphi \rangle_a \langle v, \psi \rangle_a .
\end{aligned}$$

The rest goes in the same way as in Proposition 5.6. ■

The following statement is a combination of the argument of separation of variables with the Künneth formula.

Theorem 6.3. *For the weight $a_p = p!$ we have*

$$\operatorname{spec} \Delta_r^{(a)}(X \Box Y) = \bigsqcup_{\{p,q \geq 0:\, p+q=r\}} \left(\operatorname{spec} \Delta_p^{(a)}(X) + \operatorname{spec}_q^{(a)}(Y) \right).$$

The proof goes in the same way as that of Theorem 5.8.

The weighted Hodge Laplacian $\Delta_p^{(a)}$ with the weight (6.1) is called the *normalized* Hodge Laplacian, in contrast ti the *canonical* Laplacian Δ_p.

Proposition 6.4. *Let a be the weight (6.1). Then, for any $p \geq 0$ and $n \geq 1$,*

$$\operatorname{spec} \Delta_p^{(a)}(I^n) = \left\{ (2k)_{\binom{n}{p}\binom{n-p}{k-p}} \right\}_{k=p}^{n}. \tag{6.2}$$

In particular,

$$\operatorname{spec} \Delta_1^{(a)}(I^n) = \left\{ (2k)_{n\binom{n-1}{k-1}} \right\}_{k=1}^{n}. \tag{6.3}$$

6.2 Some spectral properties of Hodge Laplacian

In this subsection let $\{\Omega_p\}_{p \geq 0}$ be any chain complex with a boundary operator ∂ and any inner product $\langle \cdot, \cdot \rangle$. Let ∂^* be the adjoint of ∂ with respect to $\langle \cdot, \cdot \rangle$. Denote

$$D'_p = \partial \partial^* |_{\Omega_p} \quad \text{and} \quad D''_p = \partial^* \partial |_{\Omega_p}$$

so that

$$\Delta_p = D'_p + D''_p.$$

These operators are shown on the diagram:

$$\begin{array}{ccccccccc}
\dots & \overset{\partial}{\leftarrow} & \Omega_{p-1} & \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} & \Omega_p & \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} & \Omega_{p+1} & \overset{\partial}{\leftarrow} & \dots \\
 & & & & & & & & , \\
 & & & \uparrow & & \uparrow & & & \\
 & & & D''_p & & D'_p & & &
\end{array}$$

Observe that both D'_p and D''_p are also non-negative definite self-adjoint operators in Ω_p.
For any $\lambda > 0$ consider the following subspaces of Ω_p:

$$\begin{aligned}
E_p(\lambda) &= \{\varphi \in \Omega_p : \Delta_p \varphi = \lambda\varphi\}, \\
E'_p(\lambda) &= \{\varphi \in \Omega_p : D'_p \varphi = \lambda\varphi\}, \\
E''_p(\lambda) &= \{\varphi \in \Omega_p : D''_p \varphi = \lambda\varphi\}.
\end{aligned}$$

We will use the following known facts:

$$E_p(\lambda) = E'_p(\lambda) \oplus E''_p(\lambda)$$

and

$$E''_p(\lambda) \cong E'_{p-1}(\lambda),$$

where the linear isomorphism is given by ∂ with inverse ∂^*.
It follows that

$$\begin{aligned}
|E_p(\lambda)| &= \left|E'_p(\lambda)\right| + \left|E''_p(\lambda)\right| \\
&= \left|E'_{p-1}(\lambda)\right| + \left|E''_{p+1}(\lambda)\right|.
\end{aligned}$$

Since the multiplicity of λ as eigenvalue of Δ_p is equal to $|E_p(\lambda)|$, it follows that

$$\operatorname{spec}_+ \Delta_p = \operatorname{spec}_+ D'_p \sqcup \operatorname{spec}_+ D''_p \tag{6.4}$$

and

$$\operatorname{spec}_+ \Delta_p = \operatorname{spec}_+ D'_{p-1} \sqcup \operatorname{spec}_+ D''_{p+1}. \tag{6.5}$$

6.3 Spectrum of the canonical Hodge Laplacian on cubes

Here we finally compute the spectrum of the canonical Hodge Laplacian Δ_p on n-cube.

Theorem 6.5. *For any $n \geq 1$, we have*

$$\operatorname{spec} \Delta_1(I^n) = \left\{(2k)_{\binom{n}{k}}\right\}_{k=1}^{n} \sqcup \left\{k_{(k-1)\binom{n}{k}}\right\}_{k=2}^{n}. \tag{6.6}$$

In particular,

$$\lambda_{\max}(\Delta_1(I^n)) = (2n)_1$$

and

$$\lambda_{\min}(\Delta_1(I^n)) = 2_{\binom{n+1}{2}}.$$

In particular, $\operatorname{spec}\Delta_1(I^n)$ contains all even numbers from 2 to $2n$, and all odd numbers from 3 to n as was conjectures.

Outline of the proof. Let $G = I^n$. Observe first that $\operatorname{spec}\Delta_1 > 0$ as $\beta_1 = 0$. In order to compute the positive spectrum $\operatorname{spec}_+\Delta_1$, we apply the identity (6.5) for the canonical Δ_1 as well as to the normalized operator $\Delta_1^{(a)}$ (with weight $a_p = p!$). This identity for $\Delta_p^{(a)}$ becomes

$$\operatorname{spec}_+\Delta_p^{(a)} = p\operatorname{spec}_+ D'_{p-1} \sqcup (p+1)\operatorname{spec}_+ D''_{p+1}. \tag{6.7}$$

In particular, for $p = 1$ we obtain from (6.5) and (6.7) that

$$\begin{aligned}\operatorname{spec}\Delta_1 = \operatorname{spec}_+\Delta_1 &= \operatorname{spec}_+ D'_0 \sqcup \operatorname{spec}_+ D''_{p+1}\\ &= \operatorname{spec}_+\Delta_0 \sqcup \operatorname{spec}_+ D''_2,\end{aligned}$$

and

$$\begin{aligned}\operatorname{spec}_+\Delta_1^{(a)} &= \operatorname{spec}_+ 1\cdot D'_0 \sqcup 2\cdot\operatorname{spec}_+ D''_2\\ &= \operatorname{spec}_+\Delta_0 \sqcup 2\operatorname{spec}_+ D''_2.\end{aligned}$$

Since

$$\operatorname{spec}_+\Delta_1^{(a)} = \operatorname{spec}\Delta_1^{(a)} = \left\{(2k)_{n\binom{n-1}{k-1}}\right\}_{k=1}^n,$$

and

$$\operatorname{spec}_+\Delta_0 = \operatorname{spec}_+\Delta_0^{(a)} = \left\{(2k)_{\binom{n}{k}}\right\}_{k=1}^n,$$

we compute $\operatorname{spec}_+ D''_2$ as follows:

$$\begin{aligned}2\operatorname{spec}_+ D''_2 &= \operatorname{spec}_+\Delta_1^{(a)} \setminus \operatorname{spec}_+\Delta_0\\ &= \left\{(2k)_{n\binom{n-1}{k-1}-\binom{n}{k}}\right\}_{k=1}^n = \left\{(2k)_{(k-1)\binom{n}{k}}\right\}_{k=2}^n.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{spec}\Delta_1 &= \operatorname{spec}_+\Delta_0 \sqcup \operatorname{spec}_+ D''_2\\ &= \left\{(2k)_{\binom{n}{k}}\right\}_{k=1}^n \sqcup \left\{k_{(k-1)\binom{n}{k}}\right\}_{k=2}^n.\end{aligned}$$

■

Theorem 6.6. *We have, for any $n \geq 2$,*

$$\operatorname{spec}\Delta_2(I^n) = \left\{\left(\frac{2k}{3}\right)_{\binom{k-1}{2}\binom{n}{k}}\right\}_{k=3}^n \sqcup \left\{k_{(k-1)\binom{n}{k}}\right\}_{k=2}^n.$$

In particular,

$$\lambda_{\max}\left(\Delta_2(I^n)\right) = n_{n-1}$$

and

$$\lambda_{\min}\left(\Delta_2(I^n)\right) = 2_{\binom{n+1}{3}}.$$

Outline of the proof. In order to compute the positive spectrum $\operatorname{spec}_+ \Delta_1$, we apply the identity (6.4) for the canonical Δ_1 as well as to the normalized operator $\Delta_1^{(a)}$ (with weight $a_p = p!$). That identity for $\Delta_p^{(a)}$ becomes

$$\operatorname{spec}_+ \Delta_p^{(a)} = (p+1) \operatorname{spec}_+ D_p' \sqcup p \operatorname{spec}_+ D_p''. \tag{6.8}$$

For $p = 2$ we obtain from (6.4) and (6.8) that

$$\operatorname{spec}_+ \Delta_2 = \operatorname{spec}_+ D_2' \sqcup \operatorname{spec}_+ D_2''$$

and

$$\operatorname{spec}_+ \Delta_2^{(a)} = 3 \operatorname{spec}_+ D_2' \sqcup 2 \operatorname{spec}_+ D_2''.$$

It follows that

$$\begin{aligned}
3 \operatorname{spec}_+ D_2' &= \operatorname{spec}_+ \Delta_2^{(a)} \setminus 2 \operatorname{spec}_+ D_2'' \\
&= \left\{ (2k)_{\binom{n}{2}\binom{n-2}{k-2}} \right\}_{k=2}^{n} \setminus \left\{ (2k)_{(k-1)\binom{n}{k}} \right\}_{k=2}^{n} \\
&= \left\{ (2k)_{\binom{n}{2}\binom{n-2}{k-2} - (k-1)\binom{n}{k}} \right\}_{k=2}^{n} = \left\{ (2k)_{\binom{k-1}{2}\binom{n}{k}} \right\}_{k=3}^{n}
\end{aligned}$$

and, hence,

$$\begin{aligned}
\operatorname{spec}_+ \Delta_2 &= \operatorname{spec}_+ D_2' \sqcup \operatorname{spec}_+ D_2'' \\
&= \left\{ \left(\frac{2k}{3} \right)_{\binom{k-1}{2}\binom{n}{k}} \right\}_{k=3}^{n} \sqcup \left\{ k_{(k-1)\binom{n}{k}} \right\}_{k=2}^{n}.
\end{aligned}$$

■

Similar ideas are used to prove the following theorem.

Theorem 6.7. *For all $1 \le p \le n$ we have*

$$\operatorname{spec} \Delta_p(I^n) = \left\{ \left(\frac{2k}{p} \right)_{\binom{k-1}{p-1}\binom{n}{k}} \right\}_{k=p}^{n} \sqcup \left\{ \left(\frac{2k}{p+1} \right)_{\binom{k-1}{p}\binom{n}{k}} \right\}_{k=p+1}^{n}. \tag{6.9}$$

In particular,

$$\lambda_{\max} \left(\Delta_p(I^n) \right) = \left(\frac{2n}{p} \right)_{\binom{n-1}{p-1}}$$

and

$$\lambda_{\min} \left(\Delta_p(I^n) \right) = 2_{\binom{n+1}{p+1}}.$$