Fusion systems and self equivalences of p-completed classifying spaces of finite groups of Lie type

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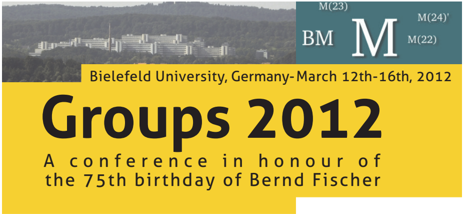

Outline

1. *Groups, classifying spaces, and fusion systems*

2. *Equivalences between fusion systems of finite groups of Lie type at primes different from the defining characteristic*

3. *Fusion systems of finite simple groups of Lie type are tame*

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Groups, classifying spaces, and fusion systems

Fix a prime p.

$\mathcal{F}_p(G)$	$\leftarrow\!\shortmid$	G	$\mapsto$	BG	$\mapsto$	$BG^\wedge_p$
fusion system		finite group		classifying space		p-completed classifying space

Classifying spaces

Given a finite group G there is a universal contractible free G-space EG.

The classifying space of G is the space of orbits

$$BG = EG/G\,.$$

It is determined by G, up to homotopy.

Examples

- $G = \mathbb{Z}/2$, $E\mathbb{Z}/2 \simeq S^\infty$ and $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$.
- G discrete, $BG \simeq K(G,1)$. Determined by $\pi_i(K(G,1)) = \begin{cases} G & \text{if } i=1, \\ 0 & \text{otherwise.} \end{cases}$

There are different ways to construct BG:

- For a discrete group G, take a wedge of circles indexed by a set of generators of the group, then attach 2-cells corresponding to the relations, so that the fundamental group of the complex is G, then attach higher cells to kill all higher homotopy groups. The resulting complex is BG.
- Bar construction is a functorial construction of BG. A homomorphism $\varphi\colon G \longrightarrow H$ induces a continuous map $B\varphi\colon BG \longrightarrow BH$.

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p-equivalences and p-completion

Fix a prime p.

Two spaces X and Y are p-equivalent if there is a 3rd space Z and maps

$$X \longrightarrow Z \longleftarrow Y$$

that induce isomorphisms in cohomology with coefficients in $\mathbb{F}_p$.

Bousfield-Kan p-completion is a coaugmented functor $\ell_X\colon X \longrightarrow X_p^\wedge$ that turns p-equivalences into homotopy equivalences.

That is, a map $f\colon X \longrightarrow Y$ induces and isomorphism in mod p cohomology

$$f^*\colon H^*(Y;\mathbb{F}_p) \xrightarrow{\cong} H^*(X;\mathbb{F}_p)$$

if and only if it induces a homotopy equivalence after p-completion:

$$f_p^\wedge\colon X_p^\wedge \xrightarrow{\simeq} Y_p^\wedge$$

Example:

- For $S^1 \simeq B\mathbb{Z}$, we have $(S^1)_p^\wedge \simeq B\mathbb{Z}_p$ and the coaugmentation $\ell_{B\mathbb{Z}}\colon B\mathbb{Z} \longrightarrow B\mathbb{Z}_p$ is given by the inclusion of the integers in the p-adic integers.

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The p-completion of BG

1. If P is a finite p-group, then BP is p-complete.
2. In general, BG is not p-complete:
   - $\pi_1(BG^\wedge_p) \cong G/O^p(G)$, where O^pG is the maximal normal p-perfect subgroup of G.
   - The universal cover is $BO^p(G)^\wedge_p$. Usually, it carries a rich higher homotopy structure, with non-trivial homotopy groups in arbitrarily large dimensions.

Example:

- Fix an odd prime p. Form the semidirect product $\mathbb{Z}/p^r \rtimes \mathbb{Z}/2$. Then:

$$\Omega\big(B(\mathbb{Z}/p^r \rtimes \mathbb{Z}/2)\big)^\wedge_p \simeq S^3\{p^r\}$$

(homotopy fibre of the degree p^r self map of S^3.) and therefore that $\big(B(\mathbb{Z}/p^r \rtimes \mathbb{Z}/2)\big)^\wedge_p$ supports the p-primary part of the homotopy groups of S^3.

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Fusion system of a finite group

Definition

Let G be a finite group, fix a prime p and $S \in \mathrm{Syl}_p(G)$, then the fusion system of G, $\mathcal{F}_p(G)$, is a category with

- *Objects: $P \leq S$, the subgroups of S, and*
- *Morphisms:*

$$\hom_{\mathcal{F}_p(G)}(P,Q) = \{\varphi\colon P \to Q \mid \exists g \in G, \varphi(x) = gxg^{-1}\} \cong N_G(P,Q)/C_G(P)$$

Fusion preserving isomorphisms

If H is another finite group we will say that the fusion systems of G and H are equivalent:

$$\mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$$

if there is $R \in \mathrm{Syl}_p(H)$ and an isomorphism $f\colon S \longrightarrow R$ that preserves fusion:

$$\varphi \in \hom_{\mathcal{F}_p(G)}(P,Q) \Longrightarrow (f|_Q) \circ \varphi \circ (f|_P)^{-1} \in \mathrm{Hom}_{\mathcal{F}_p(H)}(f(P), f(Q))$$

Martino-Priddy conjecture

Fix a prime p.

$\mathcal{F}_p(G)$	$\leftarrow$	G	$\mapsto$	BG	$\mapsto$	$BG^\wedge_p$	$\mapsto$	$\mathcal{F}_p(G)$
fusion system		finite group		classifying space		p-completed classifying space		fusion system

Martino-Priddy: The fusion system can be reconstructed from the p-completed classifying space. Given finite groups G and H, if $BG^\wedge_p \simeq BH^\wedge_p$ then $\mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$.

M-P conjecture (1996):

$$BG^\wedge_p \simeq BH^\wedge_p \iff \mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$$

Oliver (2004, 2006): M-P conjecture is true. (The proof depends on the classification of finite simple groups.)

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Equivalences between fusion systems of finite groups of Lie type at primes different from the defining characteristic

Theorem (B-Møller-Oliver)

Let $\mathbb{G}$ be a connected reductive group scheme over $\mathbb{Z}$. Fix a prime p, and finite fields $\mathbb{F}_q$, $\mathbb{F}_{q'}$ of char $\neq p$, then:

(a) $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ *if* $\overline{\langle q \rangle} = \overline{\langle q' \rangle} \leq \mathbb{Z}_p^\times$

(b) $\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p({}^\tau\mathbb{G}(q'))$ *if* $\mathbb{G} = A_n, D_n, E_6$, τ *a graph automorphism, and* $\overline{\langle q \rangle} = \overline{\langle q' \rangle}$.

(c) *In case the Weyl group of $\mathbb{G}$ contains an element which acts on the maximal torus by inverting all elements: ψ^{-1}, then*

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provided $\overline{\langle -1, q \rangle} = \overline{\langle -1, q' \rangle} \leq \mathbb{Z}_p^\times$.

(d) *For $\mathbb{G}$ of type A_n, D_n with n odd, or E_6, and τ a graph automorphism of order 2,*

$$\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q')) \quad \text{if} \quad \overline{\langle -q \rangle} = \overline{\langle q' \rangle} \leq \mathbb{Z}_p^\times$$

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i.e., $\begin{cases} \mathrm{ord}_p(q) = \mathrm{ord}_p(q') = s \text{ and } v_p(q^s - 1) = v_p({q'}^s - 1), \text{ if } p \text{ is odd,} \\ q \equiv q' \bmod 8 \text{ and } v_2(q^2 - 1) = v_2({q'}^2 - 1), \text{ if } p = 2. \end{cases}$

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Some additional cases:

- If $q \equiv 1 \bmod p$, then

$$\text{for } p \neq 3, \qquad \mathcal{F}_p(G_2(q)) \simeq \mathcal{F}_p({}^3D_4(q)), \text{ and}$$

$$\text{for } p \neq 2, \qquad \mathcal{F}_p(F_4(q)) \simeq \mathcal{F}_p({}^2E_6(q))$$

Ingridients of the proof

(A) Results by Friedlander and by Jacksowski-McClure-Oliver:

If $\mathbb{G}$ is a connected reductive group scheme over $\mathbb{Z}$ and $(p,q)=1$, there is a homotopy pull-back square

$$\begin{array}{ccc} B^{\tau}\mathbb{G}(q)^{\wedge}_p & \longrightarrow & B\mathbb{G}(\mathbb{C})^{\wedge}_p \\ \big\downarrow & & \big\downarrow \Delta \\ B\mathbb{G}(\mathbb{C})^{\wedge}_p & \xrightarrow{1\times\tau\psi^q} & B\mathbb{G}(\mathbb{C})^{\wedge}_p \times B\mathbb{G}(\mathbb{C})^{\wedge}_p \end{array}$$

where ψ^q is the unstable Adams map of exponent q and τ a graph automorphism. $(B\mathbb{G}(\mathbb{C})^{\wedge}_p \simeq BG^{\wedge}_p$, G the unitary form of $\mathbb{G}(\mathbb{C})$.)

The case $\mathbb{G}=GL$ was first considered by Quillen in order to compute $H^*(GL_n(q),\mathbb{F}_p)$. (Related to his work on K-theory of finite fields.)

Ingridients of the proof

(B) Homotopy fixed points: $B^{\tau}\mathbb{G}(q)^{\wedge}_{p} \simeq (BG^{\wedge}_{p})^{h\langle \tau\psi^{q}\rangle}$.

In a homotopy pull-back diagram:

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{1\times\alpha} & X \times X \end{array}$$

we can interpret

$$E \simeq X^{h\alpha} = \mathrm{Map}_{\mathbb{Z}}(\mathbb{R}, X)\,,$$

the space of homotopy fixed points, after rigidifying the action of $\mathbb{Z}$ on X given by α.

Furthermore, $X^{h\alpha} \simeq \Gamma(X_{h\alpha} \downarrow S^1)$ is a space of sections of the fibre bundle:

$$X \longrightarrow X_{h\alpha} \longrightarrow S^1$$

where $X_{h\alpha} = X \times I/\sim$, $(0,x) \sim (1,\alpha(x))$, is the mapping torus.

Ingridients of the proof

Key observation: Sometimes $X \longrightarrow X_{h\alpha} \longrightarrow S^1 (\simeq B\mathbb{Z})$ extends to a fibration

$$X_p^\wedge \longrightarrow (X_{h\alpha})_p^\wedge \longrightarrow (S^1)_p^\wedge\ (\simeq B\mathbb{Z}_p)$$

thus, the action of $\mathbb{Z}$ generated by α extends to an action of $\mathbb{Z}_p$.

Theorem

Fix a prime p, X a connected and p-complete space satisfying

- $H^*(X, \mathbb{F}_p)$ *Noetherian*
- $\operatorname{Out}(X)$ *detected* $\hat{H}^*(X, \mathbb{Z}_p) := \varprojlim H^*(X, \mathbb{Z}/p^k)$.

Let α, β be self equivalences of X with $\overline{\langle\alpha\rangle} = \overline{\langle\beta\rangle}$ in $\operatorname{Out}(X)$, then

$$X^{h\alpha} \simeq X^{h\beta}.$$

Here, $\operatorname{Out}(X)$ is given the p-adic topology determined by the basis of open neighborhoods of the identity:

$$U_k = \{[f] \in \operatorname{Out}(X) | f^* = \mathrm{id} \text{ on } H^*(X, \mathbb{Z}/p^k\}$$

Abstract fusion systems

Definition (Puig)

A fusion system $\mathcal{F}$ over a finite p-group S consists of a set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for every pair P,Q of subgroups of S such that

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$$

and form a category where every morphism decomposes as an isomorphism followed by an inclusion.

It is saturated if it satisfies some extra axioms. ▸ Axioms

Definition

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Known examples of exotic fusion systems

1. There is only one known family of exotic fusion systems at the prime 2: $Sol(q)$, q an odd prime power.

   $Sol(q)$ are saturated fusion systems studied by Solomon, later by Benson, and formalized by Levi-Oliver, defined over the Sylow 2-subgroup of $Spin(7, q)$.

2. [Ruiz-Viruel] Classification of saturated fusion systems defined over the extraspecial groups of order p^3 and exponent p, (p odd prime): There are exactly 3 exotic examples at the prime 7.

3. [Diaz-Ruiz-Viruel] Complete the classification of saturated fusion systems over finite p-groups of rank 2. New exotic examples appear at the prime 3.

4. [B-Møller] Construction of classifying spaces of new exotic examples: $BX(m,r,n)(q)$ $n \geq p$ and $r > 2$, $BX_{29}(q)$ $p = 5$, $q \equiv 1 \bmod p$, and $BX_{34}(q)$, $p = 7$ and $\equiv 1 \bmod p$.

5. [Ruiz] Construction of the exotic saturated fusion systems $X(m,r,n)(q)$ as subsystems of index prime to p in the fusion system of the general linear groups $GL_{mn}(\mathbb{F}_q)$.

Known examples of exotic fusion systems

1. There is only one known family of exotic fusion systems at the prime 2: $Sol(q)$, q an odd prime power.

   $Sol(q)$ are saturated fusion systems studied by Solomon, later by Benson, and formalized by Levi-Oliver, defined over the Sylow 2-subgroup of $Spin(7, q)$.

2. [Ruiz-Viruel] Classification of saturated fusion systems defined over the extraspecial groups of order p^3 and exponent p, (p odd prime): There are exactly 3 exotic examples at the prime 7.

3. [Diaz-Ruiz-Viruel] Complete the classification of saturated fusion systems over finite p-groups of rank 2. New exotic examples appear at the prime 3.

4. [B-Møller] Construction of classifying spaces of new exotic examples: $BX(m, r, n)(q)$ $n \geq p$ and $r > 2$, $BX_{29}(q)$ $p = 5$, $q \equiv 1 \bmod p$, and $BX_{34}(q)$, $p = 7$ and $\equiv 1 \bmod p$.

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Tame systems

Definition

A saturated fusion system $\mathcal{F}$ over a finite p-group S is tame if there exists a finite group G such that

(i) $\mathcal{F} \simeq \mathcal{F}_p(G)$, *(realizable).*

(ii) *The natural map*

$$\kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG_p^\wedge)$$

is split surjective (tamely realized by G).

[Andersen-Oliver-Ventura] Fusion systems that are not tamely realized are reductions of exotic systems.

This has to be made precise by defining *reduced* fusion systems and a *reduction process*. This is based in know extension theory for saturated fusion systems.

Theorem (Joint work in progress with J. Møller and B. Oliver)

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End

Saturation Axioms for fusion systems

Let $\mathcal{F}$ be a fusion system over a p-group S.

1. A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.
2. A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

Definition

A fusion system $\mathcal{F}$ over a p-group S is a saturated if the following two conditions hold:

(I) For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p \mathrm{Aut}_{\mathcal{F}}(P)$.

(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

Return