Platonic polygonal complexes

Ian Leary (University of Southampton)

16th March 2012

For the 75th birthday of Bernd Fischer

Joint with T Januszkiewicz, R Valle and R Vogeler

[These slides were not used at the conference, but were prepared later to contribute to the conference website.]

Platonic polygonal complexes

A *polygonal complex* is a union of polygons such that the intersection of any two polygons is either empty or a vertex of each of them or an edge of each of them. A polygonal complex carries a topology in which a set is open if and only if its intersection with each polygon is open.

A *flag* in a polygonal complex is mutually incident triple, consisting of a vertex, an edge, and a polygon.

A polygonal complex is *platonic* if it admits a flag-transitive group of symmetries. Examples include the surface of each platonic solid, and the regular tesselation of the plane by squares.

Degenerate cases

Sometimes we allow more general complexes: consider the torus built from a 2×2-array of squares by identifying opposite sides. This is platonic, but it isn't quite a polygonal complex because the intersections of pairs of squares are wrong: some pairs intersect in two opposite edges; others intersect in their four corners.

More degenerate cases

Even worse is the torus built from a single square; here the boundary of the square has self-intersections although the interior of the square maps injectively into the torus. This torus has one vertex, two edges and one square, but we would like it to have 8 flags.

For a complex like this torus, we modify the definition of a flag: a flag is the image inside the complex of one of the $2d$ triangles into which a d-gon is cut by its axes of symmetry.

Rank 3 incidence geometries with specified 0-, 1- and 2-residues are closely related to polygonal complexes: this notion includes polygonal complexes and the first type of degenerate complex but not the second.

Vertex links

We aim to classify simply connected platonic polygonal complexes with specified vertex links. In a platonic complex all polygons have the same number, d, of sides, and all vertices have isomorphic link graphs. The links to which our methods apply are the following simple graphs Γ:

Any Γ with the property that the natural map

$$\mathrm{Aut}(\Gamma, v) \to \mathrm{Symm}(N_v)$$

is an isomorphism;

The graph O_n which is the 1-skeleton of the unit ball in the 1-norm on $\mathbb{R}^n$, or equivalently the multipartite graph with n parts of size 2.

Related work

Ballmann and Brin classified polygonal complexes satisfying a stronger symmetry condition than our 'platonic', for any link graph satisfying the first of our two conditions.

Świątkowski classified platonic polygonal complexes in which each edge is contained in precisely three polygons.

Curvature

If d (the number of sides of the polygons) is at least 6, then every platonic d-gonal complex admits a CAT(0)-metric.

For $d \geq 6$ all platonic polygonal complexes are infinite.

In contrast, one can show that for the complete graph K^n as vertex link, all $(n, 4)$-complexes are finite, and for fixed n, the $(n, 4)$-complex with the largest diameter is the 2-skeleton of the n-cube.

$(n, 5)$-complexes can be either finite or infinite.

To simplify things, let's just consider the case when the link graph is K^n from now on. *[I did not do this in the actual lecture.]*

Holonomy

For each n and d there is at least one (n, d)-complex, coming from the Coxeter group whose Dynkin diagram is a line with n nodes and one terminal edge labeled 'd'.

This one has trivial 'holonomy': the faces which share an edge with any given face form $n - 2$ disjoint rings around the face.

In general, if you stand on a face, and walk around its perimeter with your hand on a neighbouring face, your hand might not come back to the same place after one full circle. The permutation of the neighbours that arises this way is the *holonomy* of the complex.

Invariants

Let G denote the full group of symmetries of an (n, d)-complex.

The vertex stabilizer G_v can be identified with a group of permutations of N_v, the set of n neighbours of v.

If e is the edge from v to w, the edge stabilizer G_e acts on the set $N_v - \{w\}$ and on the set $\{v, w\}$. Together these two actions are faithful.

The holonomy going around the face containing the segment (u, v, w) can be viewed as a permutation of N_v, which preserves the subset $\{u, w\}$.

Classification

It turns out that these invariants classify (n, d)-complexes:

For $d \geq 6$, there is a bijection between isomorphism types of (n, d)-complexes and conjugacy classes of maximal triples of the form:

$$(G_v, \bar{G}_e, \Phi).$$

Here, G_v is a 2-transitive subgroup of S_n;

$\bar{G}_e$ is an index 1 or 2 supergroup of the 1-point stabilizer in G_v;

Φ is a permutation of n points having certain properties.

Properties of Φ

Let r be an element of the 2-set stabilizer in G_v that is not in the 2-point stabilizer.

Let s be an element of $\bar{G}_e$ that is not in G_v (or let $s = 1$ if there is no such element).

Φ is centralized by the 2-point stabilizer in G_v

Φ is inverted by r

Φ is inverted by s

$(rs)^d \Phi^{-1}$ is an element of the 2-point stabilizer in G_v.

Corollaries

There is exactly one (n, d)-complex with trivial holonomy, coming from the triple $(S_n, S_{n-1}, (n-1, n)^d)$.

For 'most' values of n, there are no other (n, d)-complexes.

The classification of (n, d)-complexes depends only on the residue of d modulo $n!$.

If d is odd, then one can find a sequence $n = n_i$ of values for which there are arbitrarily many distinct (n, d)-complexes.

Small values of n

$n = 4$:
$(S_4, S_3, (1,2))$ for any d.

$n = 5$:
$(A_5, S_4, (1,2,3))$ for any d
$(AGL(1,5), D_8, (a,b,c))$ for d not divisible by 3
$(AGL(1,5), GL(1,5), (a,b))$ for d odd

$n = 6$:
$(PSL(2,5), ASL(1,5), (a,b))$ for d odd
$(PGL(2,5), AGL(1,5), (a,b)(c,d))$ for d even

$n = 7$:
$(AGL(1,7), GL(1,7), (a,b)(c,d))$ for d odd
$(AGL(1,7), D_{12}, (a,b,c))$ for d not divisible by 3
$(GL(3,2), AGL(2,2), (a,b)(c,d))$ for d even

[Φ acts on $\{n-1, n\}$ as $(n-1, n)^d$. This factor is omitted in the table.]

Example: S_n, A_n

The center of S_{n-2} is trivial for $n > 4$. So the only complexes with $G_v = S_n$ are the untwisted ones $(S_n, S_{n-1}, (n-1, n)^d)$ and the Ballmann-Brin example $(S_4, S_3, (3,4)^d(1,2))$, which exist for all d.

The center of A_{n-2} is trivial for $n > 5$. So the only extra complex with $G_v = A_n$ is $(A_5, S_4, (1,2,3)(4,5))$, which exists for all d.

Example: $AGL(1,p)$

Let $G_v = AGL(1,p)$ acting on p points. The stabilizer of 0 is $GL(1,p)$. The 2-point stabilizer is trivial. Choices for $\bar{G}_e$ correspond to integers m such that $x \mapsto x^m$ is an involution on the elements of $\mathbb{F}_p$.

If $p-1$ is divisible by 8 and by k distinct odd primes, then there are 2^{k+2} choices for m.

For each such m, one obtains a triple $(AGL(1,p), \bar{G}_e, \Phi)$. The permutation Φ is the dth power of $x \mapsto 1 - x^m$, and so this gives a new (p,d)-complex for each d not dividing the order of the map $x \mapsto 1 - x^m$, viewed as a self-map of $\mathbb{F}_p - \{0,1\}$.

Example: $PGL(2, p)$

Let $G_v = PGL(2, p)$ acting on $p + 1$ points. The stabilizer of ∞ is $AGL(1, p)$. The stabilizer of the points $0, \infty$ is $GL(1, p)$. The map $x \mapsto 1/x$ is a choice for r, and $s = \mathrm{Id}$ is the only choice. Hence Φ has to have order 2.

End up with just one nontrivial choice for Φ: the map $x \mapsto -x$ which only works if d is even.

$G_v = PSL(2, p)$ gives just one more case: if $p = 5$ and d is odd, the map $x \mapsto 1/x$ will work for Φ. (This only works for $p = 5$ because only then does $x \mapsto 1/x$ commute with the action of the index two subgroup of $GL(1, p)$.)

Some other examples

Taking G_v to be the Higman-Sims simple group acting on 176 points gives an example for any odd d.

Taking G_v to be the Conway group acting on 276 points doesn't give any examples.

The least n for which there are no examples with nontrivial holonomy for any d is $n = 21$.