Toy examples concerning the Bousfield lattice

Greg Stevenson

October 26, 2011

Contents

1 Introduction

The aim of this note is to give some examples of toy tensor triangulated categories to illustrate certain properties of the Bousfield lattice; it serves to show that one can not be too ambitious in trying to prove general results about the Bousfield lattice.

2 Recollections on Heyting and Boolean algebras

In this section we give a brief reminder of the definitions of Heyting algebras, Boolean algebras, and of the Booleanization functor.

Notation 2.1. Let L be a lattice. All our lattices will be bounded and we denote by 0 and 1 (or 0_L and 1_L if our notation would otherwise be ambiguous) the minimal and maximal elements of L respectively. We use $\vee$ and $\wedge$ to denote the join and meet on L.

Definition 2.2. A *Heyting algebra* is a lattice H equipped with an implication operation $\Rightarrow : H^{\mathrm{op}} \times H \longrightarrow H$ satisfying the following universal property:

$$(l \wedge m) \leq n \text{ if and only if } l \leq (m \Rightarrow n)$$

for all $l, m, n \in H$. A *complete Heyting algebra* is a Heyting algebra which is complete as a lattice.

A morphism of Heyting algebras is a morphism of lattices which preserves implication. A morphism of complete Heyting algebras is a morphism of lattices which preserves implication and arbitrary joins.

It is standard that Heyting algebras are distributive and that complete Heyting algebras are the same as frames i.e., complete distributive lattices. In a frame L the implication operation $l \Rightarrow m$, for $l, m \in L$, is given by the join of the set $\{x \in L \mid x \wedge l \leq m\}$. However, not every morphism of frames preserves the implication operation. We denote the category of (complete) Heyting algebras and (complete) Heyting algebra morphisms by Heyt (cHeyt).

Definition 2.3. Let H be a Heyting algebra. The *pseudocomplement* or *negation* of $h \in H$ is defined to be

$$\neg h := (h \Rightarrow 0).$$

The element $\neg h$ is the largest element of H such that $h \wedge \neg h = 0$. We say h is *regular* if $\neg\neg h = h$ and that h is *complemented* if $h \vee \neg h = 1$. We recall that regularity of h is equivalent to the condition that there exists h' with $h = \neg h'$ and that every complemented element of a Heyting algebra is regular.

We denote by $H_{\neg\neg}$ the subposet (in general it is not a subalgebra) of regular elements of H.

Definition 2.4. A (complete) Heyting algebra B is a *(complete) Boolean algebra* if it satisfies one (and hence both) of the following equivalent conditions:

1 every element of B is regular;

2 every element of B is complemented.

A morphism of (complete) Boolean algebras is just a morphism of (complete) Heyting algebras.

We denote the category of (complete) Boolean algebras and morphisms of (complete) Boolean algebras by Bool (cBool).

Let H be a Heyting algebra. Then double negation defines a morphism of Heyting algebras $\neg\neg\colon H \longrightarrow H_{\neg\neg}$. In fact double negation gives a monad on Heyt. If H is complete then so is $H_{\neg\neg}$ and double negation gives a monad on cHeyt. This is summarised in the following well known theorem.

Theorem 2.5. *Let H be a (complete) Heyting algebra. Then the poset $H_{\neg\neg}$ is a (complete) Boolean algebra called the* Booleanization *of H. Furthermore, $H \mapsto H_{\neg\neg}$ extends to a functor*

$$B\colon \mathsf{Heyt} \longrightarrow \mathsf{Bool}$$

which is left adjoint to the fully faithful inclusion $\mathsf{Bool} \longrightarrow \mathsf{Heyt}$. The double negation $H \longrightarrow H_{\neg\neg}$ is the unit of this adjunction. Moreover, the functor B restricted to cHeyt gives a left adjoint to the fully faithful inclusion $\mathsf{cBool} \longrightarrow \mathsf{cHeyt}$.

3 Any complete Boolean algebra is a Bousfield lattice

In this section we show how to construct, starting from a frame L, an algebraic tensor triangulated category $\mathcal{T}_L$ whose Bousfield lattice is the Booleanization of L. Hence if L is a complete Boolean algebra then the Bousfield lattice of $\mathcal{T}_L$ is precisely L. Let us explain the construction. We fix a complete distribute lattice i.e., a frame, L and a field k. We define a triangulated category $\mathcal{T}_L$ by

$$\mathcal{T}_L = \prod_{l\in L\setminus\{0_L\}} \mathsf{D}(k_l)$$

where $k_l = k$ for all $l \in L \setminus 0_L$ - the subscript is merely to keep track of the entries. For a k-vector space V we use $\Sigma^i V_l$ to denote the object of $\mathcal{T}_L$ which is $\Sigma^i V$ in the lth position and zero elsewhere. With this convention an object $(X_l)_{l\in L\setminus\{0\}}$ of $\mathcal{T}$ is $\coprod_l X_l$. The triangulation is just given levelwise by the usual triangulated structure on the unbounded derived category of k-vector spaces $D(k)$. It is easily seen that $\mathcal{T}_L$ is compactly generated, pure semisimple, and has a stable combinatorial model. For convenience we will often denote the zero object 0 of $\mathcal{T}_L$ by k_{0_L} and think of it as the "generator corresponding to 0_L" (thus we do not worry about excluding 0_L when indexing generators of $\mathcal{T}_L$ over L).

We now show how to use the lattice structure of L to define an exact symmetric monoidal structure on $\mathcal{T}_L$. For $l, l' \in L$ we set

$$k_l \otimes k_{l'} = k_{l\wedge l'}$$

where we use the identification $k_{0_L} = 0$. This extends to a monoidal structure on $\mathcal{T}_L$ which is exact and coproduct preserving in each variable with unit k_1. Indeed, every object of $\mathcal{T}_L$ is a sum of suspensions of the k_l for $l \in L$ so the rule above, together with the usual tensor product on $\mathsf{D}(k)$, determines (an essentially unique) exact coproduct preserving extension.

Let us denote the Bousfield lattice of $\mathcal{T}_L$, our main object of interest, by $A(\mathcal{T}_L)$. We recall from [4] that $A(\mathcal{T}_L)$ is a set rather than a proper class, although this will become clear through explicit computation. For an object X of $\mathcal{T}_L$ we write $A(X)$ for its Bousfield class. The set $A(\mathcal{T}_L)$ has a natural structure of complete lattice where the Bousfield classes are ordered by reverse inclusion and the join of the Bousfield classes $\{A(X_i) \mid i \in I\}$ is given by $A(\coprod_i X_i)$. Since it should not cause confusion we shall also use $\wedge$ and $\vee$ to denote the meet and join of the Bousfield lattice.

We begin with the simple observation that every Bousfield class in $\mathcal{T}_L$ comes from one of the standard generators k_l. Before proving this we note that, for any object X of $\mathcal{T}_L$, the class $A(X)$ is determined by the k_l it contains; this is immediate as Bousfield classes are closed under summands and every object of $\mathcal{T}_L$ is a sum of suspensions of the k_l.

Lemma 3.1. *Let X be an object of $\mathcal{T}_L$. Then there exists an $l \in L$ such that $A(X) = A(k_l)$.*

Proof. We can write X as a coproduct of suspensions of the k_m for $m \in L$ and, since suspension does not change Bousfield classes, we may without loss of generality take $X \cong \coprod_i k_{l_i}$ where $l_i \in L$ are lattice elements indexed by some set I. Let us write l for the join $\vee_i l_i$ of the l_i. Then

$$\begin{aligned}
A(X) &= A(\coprod_i k_{l_i}) \\
&= \langle k_m \mid (\coprod_i k_{l_i}) \otimes k_m = 0\rangle_{\mathrm{loc}} \\
&= \langle k_m \mid \coprod_i (k_{l_i} \otimes k_m) = 0\rangle_{\mathrm{loc}} \\
&= \langle k_m \mid \coprod_i (k_{l_i \wedge m}) = 0\rangle_{\mathrm{loc}} \\
&= \langle k_m \mid l_i \wedge m = 0_L \ \forall i \in I\rangle_{\mathrm{loc}} \\
&= \langle k_m \mid l \wedge m = 0_L\rangle_{\mathrm{loc}} \\
&= A(k_l).
\end{aligned}$$

□

Remark 3.2. We proved a little more than we stated in the Lemma and we wish to record it for use later. Namely, we showed that if $\{l_i\}_{i\in I}$ is a set of elements of L then $A(\coprod_i k_{l_i}) = A(k_{\vee_i l_i})$.

Next we shall describe the meet on $A(\mathcal{T}_L)$. By the last lemma it is sufficient to do this for the Bousfield classes of the k_l with $l \in L$. Given objects X and Y of $\mathcal{T}_L$ we set $A(X) \otimes A(Y) := A(X \otimes Y)$. Observe that if $X \leq X'$ and $Y \leq Y'$ then there is an inequality $A(X) \otimes A(Y) \leq A(X') \otimes A(Y')$.

Lemma 3.3. *Let l and l' be elements of L. Then the meet of $A(k_l)$ and $A(k_{l'})$ is given by $A(k_l) \otimes A(k_{l'}) = A(k_{l\wedge l'})$.*

Proof. We need to show that $A(k_{l\wedge l'})$ is the greatest lower bound for $A(k_l)$ and $A(k_{l'})$. So suppose $A(k_x) \leq A(k_l)$ and $A(k_x) \leq A(k_{l'})$. Then

$$A(k_x) = A(k_{x\wedge x}) = A(k_x) \otimes A(k_x) \leq A(k_l) \otimes A(k_{l'}) = A(k_{l\wedge l'})$$

which shows that $A(k_{l\wedge l'})$ is the greatest lower bound for $A(k_l)$ and $A(k_{l'})$ and completes the proof. □

It follows easily that $A(\mathcal{T}_L)$ is a frame (aka a complete Heyting algebra).

Lemma 3.4. *The Bousfield lattice $A(\mathcal{T}_L)$ is distributive.*

Proof. Let l be an element of L and $\{m_i\}_{i\in I}$ a set of elements of L indexed by a set I.

Then

$$\begin{aligned} A(k_l) \wedge (\bigvee_i A(k_{m_i})) &= A(k_l) \otimes A(\coprod_i k_{m_i}) \\ &= A(k_l \otimes (\coprod_i k_{m_i})) \\ &= A(\coprod_i (k_l \otimes k_{m_i})) \\ &= \bigvee_i A(k_l \otimes k_{m_i}) \\ &= \bigvee_i (A(k_l) \wedge A(k_{m_i})) \end{aligned}$$

where we have used Lemma 3.3 and Remark 3.2. □

As usual we can then define implication for $l, m \in L$ by

$$(A(k_l) \Rightarrow A(k_m)) = \bigvee_{A(k_x) \wedge A(k_l) \leq A(k_m)} A(k_x).$$

In fact $A(\mathcal{T}_L)$ is a complete Boolean algebra. Before proving this let us give a concrete description of the negation operation on the Bousfield lattice. This negation operator is a general form of the one originally considered by Bousfield [2] (also see [3]).

Lemma 3.5. *For $l \in L$ there is an equality*

$$\neg A(k_l) = A(\coprod_{k_m \in A(k_l)} k_m) = A(k_{\neg l}).$$

Proof. Recall that $\neg A(k_l)$ is, by definition, $A(k_l) \Rightarrow A(k_{0_l})$ where $A(k_{0_L}) = \mathcal{T}$. Explicitly we have

$$\neg A(k_l) = \bigvee_{A(k_x) \wedge A(k_l) \leq A(k_{0_L})} A(k_x).$$

Now the lattice elements representing Bousfield classes in the indexing set in the wedge occurring above can be rewritten, using Lemma 3.3, as

$$\{x \in L \mid A(k_{x \wedge l}) \leq A(k_{0_L})\} = \{x \in L \mid k_{x \wedge l} = k_{0_L}\} = \{x \in L \mid x \wedge l = 0_L\}$$

which is precisely the indexing set of the join occurring in the explicit definition of $(l \Rightarrow 0_L)$. Together with Remark 3.2 this gives the claimed equalities. □

Proposition 3.6. *The Bousfield lattice of $\mathcal{T}_L$ is a complete Boolean algebra.*

Proof. We have just seen that $A(\mathcal{T}_L)$ is a frame. We will show that every element of $A(\mathcal{T}_L)$ is regular i.e., check that $\neg\neg A(k_l) = A(k_l)$ for all $l \in L$.

So let l be an element of L. Since $l \wedge \neg l = 0_L$ we have $k_l \otimes k_{\neg l} = 0$. In particular, k_l is a summand in $\coprod_{k_m \in A(k_{\neg l})} k_m$ which, by the last lemma gives an object representing $\neg\neg A(k_l)$. Thus $\neg\neg A(k_l) \subseteq A(k_l)$.

On the other hand say $k_w \otimes k_l = 0$ i.e., $w \wedge l = 0_L$. We know, by the last lemma, that $\neg A(k_l) \subseteq A(k_w)$. The double negation is given by

$$\neg(\neg A(k_l)) = A(\coprod_{k_x \in \neg A(k_l)} k_x)$$

where, by definition, each k_x lies in $\neg A(k_l)$ and hence in $A(k_w)$. Thus $k_w \in \neg\neg A(k_l)$ showing $A(k_l) \subseteq \neg\neg A(k_l)$ and completing the proof. □

We now want to compare the lattice L to $A(\mathcal{T}_L)$. Define an assignment $\phi\colon L \longrightarrow A(\mathcal{T}_L)$ by $\phi(l) = A(k_l)$. Our claim is that ϕ is a well defined morphism of complete Heyting algebras which identifies $A(\mathcal{T}_L)$ with the Booleanization of L. It is clear that ϕ is well defined and it is surjective as we have noted above that every Bousfield class is of the form $A(k_l)$.

Lemma 3.7. *The assignment $\phi\colon L \longrightarrow A(\mathcal{T}_L)$ is a morphism of frames.*

Proof. We need to check that ϕ is monotone, and preserves finite meets and infinite joins.

Let us first show that ϕ is monotone. Suppose $l \leq m$ in L. Then for all $x \in L$ we have $x \wedge l \leq x \wedge m$ and so $x \wedge m = 0_L$ implies that $x \wedge l = 0_L$. Hence $A(k_l) \supseteq A(k_m)$ i.e., $A(k_l) \leq A(k_m)$ as required.

The map ϕ preserves finite meets by Lemma 3.3. In order to show it preserves joins let $\{l_i\}_{i\in I}$ be a set of elements of L with join l. We have

$$\phi(\bigvee_i l_i) = A(k_{\vee_i l_i}) = A(\coprod_i k_{l_i}) = \bigvee_i A(k_{l_i}) = \bigvee_i \phi(l_i),$$

where the second equality is Remark 3.2. □

The next lemma shows that ϕ is in fact a morphism of complete Heyting algebras.

Lemma 3.8. *The map ϕ preserves implication.*

Proof. To start with let us recall that by Lemma 3.5 the map ϕ commutes with joins, and by Proposition 3.6 $A(\mathcal{T}_L)$ is a Boolean algebra and so negation is an involution on $A(\mathcal{T}_L)$. Thus for all $l \in L$ we have $\phi(\neg\neg l) = \phi(l)$. We need to check that

$$\begin{aligned}\phi(l \Rightarrow m) = \phi(\neg\neg(l \Rightarrow m)) = \phi((\neg\neg l) \Rightarrow (\neg\neg m)) &= \phi(\bigvee_{x \wedge (\neg\neg l) \leq (\neg\neg m)} x) \\ &= \bigvee_{x \wedge (\neg\neg l) \leq (\neg\neg m)} A(k_x)\end{aligned}$$

agrees with

$$\phi(l) \Rightarrow \phi(m) = \bigvee_{A(k_x) \wedge A(k_l) \leq A(k_m)} A(k_x).$$

Since ϕ is order preserving we have $\phi(l \Rightarrow m) \leq (\phi(l) \Rightarrow \phi(m))$; if $x \wedge l \leq m$ then $A(k_x) \wedge A(k_l) \leq A(k_m)$.

On the other hand suppose $A(k_y) \wedge A(k_l) \leq A(k_m)$ i.e., for $z \in L$ we have $m \wedge z = 0_L$ implies that $(y \wedge l) \wedge z = 0_L$. In other words there is a containment

$$\{z \in L \mid m \wedge z \leq 0_L\} \subseteq \{z \in L \mid (y \wedge l) \wedge z \leq 0_L\}$$

of the index sets defining the negations of m and $y \wedge l$. Thus $\neg m \leq \neg(y \wedge l)$ and so $\neg\neg m \geq \neg\neg(y \wedge l) = (\neg\neg y) \wedge (\neg\neg l)$. Hence $A(k_{\neg\neg y}) = A(k_y)$ occurs in the wedge defining $\phi(\neg\neg(l \Rightarrow m))$ and we see that $(\phi(l) \Rightarrow \phi(m)) \leq \phi(\neg\neg(l \Rightarrow m)) = \phi(l \Rightarrow m)$ completing the proof. □

We are now ready to show that our construction gives a rather bizarre realisation of the Booleanization of L.

Proposition 3.9. *The morphism ϕ induces an isomorphism of complete Boolean algebras $A(\mathcal{T}_L) \cong L_{\neg\neg}$.*

Proof. By the universal property of the Booleanization (see Theorem 2.5) the morphism ϕ must factor via a unique map of complete Boolean algebras $\phi' \colon L_{\neg\neg} \longrightarrow A(\mathcal{T}_L)$. Since ϕ is surjective so is ϕ'.

The map ϕ' is also injective. Indeed, $\phi'(\neg\neg l) = \phi'(\neg\neg m)$ if and only if $A(k_l) = A(k_m)$ if and only if

$$\{z \in L_{\neg\neg} \mid z \wedge (\neg\neg l) = 0_L\} = \{z \in L_{\neg\neg} \mid z \wedge (\neg\neg m) = 0_L\},$$

which can happen if and only if $\neg\neg l = \neg\neg m$. This last statement is a consequence of the fact that elements of a Boolean algebra are completely determined by their annihilators [5]. □

Corollary 3.10. *Every complete Boolean algebra occurs as the Bousfield lattice of an algebraic tensor triangulated category which can be presented as the homotopy category of a combinatorial stable monoidal model category.*

4 Not every localizing ideal is a Bousfield class

We give an example of a very simple tensor triangulated category possessing a localizing $\otimes$-ideal which is not a Bousfield class; this shows that Conjecture 9.1 of [3] need not be true in an arbitrary tensor triangulated category. One can view the example as a special case of the construction involving lattices. However, we instead choose to describe it via a similar construction using a monoid. We will use this construction later anyway and it does no harm to adopt this point of view now.

Let k be a field and let M denote the monoid with two elements $\{1, m\}$ where 1 is the identity and $m^2 = m$. Let $\mathcal{T}_M$ denote the triangulated category $\mathsf{D}(k_1) \oplus \mathsf{D}(k_m)$, where $k_1 = k_m = k$. We define a tensor product $\otimes$ on $\mathcal{T}_M$ using the monoid structure of M: k_1 is the unit object and $k_m \otimes k_m \cong k_m$. As usual one extends this using the fact that $\mathsf{D}(k)$ is pure-semisimple with unique indecomposable object, up to suspension, k to obtain a tensor triangulated structure on $\mathcal{T}_M$. It is easily checked that $\mathcal{T}_M$ is a compactly generated algebraic tensor triangulated category arising as the homotopy category of a stable combinatorial monoidal model category i.e., it is an exceedingly nice triangulated category.

It is clear that $A(\mathcal{T}_M)$ is the two element lattice consisting of $A(0)$ and $A(k_1)$. On the other hand $\langle k_m \rangle_{\mathrm{loc}}$ is a non-zero proper $\otimes$-ideal of $\mathcal{T}_M$; there is no object X of $\mathcal{T}_M$ such that $\langle k_m \rangle_{\mathrm{loc}} = A(X)$.

The category $\mathcal{T}_M$ also gives an example of a smashing localization where the acyclization functor is given by tensoring with an idempotent object but the localization functor is not. In particular, $\mathcal{T}_M$ cannot be rigidly compactly generated i.e., the full subcategory of compact objects is not rigid. To see this note that $\langle k_m \rangle_{\mathrm{loc}}$ is, as noted above, a localizing $\otimes$-ideal generated by the compact object k_m. It is thus smashing and one sees easily that the corresponding acyclization functor is given by $k_m \otimes (-)$. However, there is no tensor idempotent realizing the localization functor, given by projection onto the component $D(k_1)$, as it would have to be tensor orthogonal to k_m and no such non-zero object exists in $\mathcal{T}_M$.

5 Strange behaviour of the Bousfield lattice under localization

We now give a slightly modified version of the last example which shows that the Bousfield lattice is not necessarily well behaved under localization by a tensor ideal; in particular we show that localizing by the nilradical can destroy the good behaviour of the Bousfield lattice. To begin, let us define what me mean by the nilradical of a tensor triangulated category.

Definition 5.1. Let $\mathcal{T}$ be a tensor triangulated category with small coproducts. The *nilradical* of $\mathcal{T}$, denoted $\sqrt{\mathcal{T}}$, is the smallest radical localizing $\otimes$-ideal containing 0. Explicitly, it is the smallest localizing subcategory of $\mathcal{T}$ which is closed under tensoring with all objects of $\mathcal{T}$ and has the property that if it contains $Y^{\otimes n}$ then it contains Y.

We will use the same construction as in the last section. Let M be the commutative monoid $\{0, 1, x, y\}$ with multiplication table

	0	1	x	y
0	0	0	0	0
1	0	1	x	y
x	0	x	0	0
y	0	y	0	y

As in the last section we associate to M a tensor triangulated category $\mathcal{T}_M = \mathsf{D}(k_1) \oplus \mathsf{D}(k_x) \oplus \mathsf{D}(k_y)$ with tensor product defined using the multiplication of M. One checks easily that the spectrum of localizing prime tensor ideals of $\mathcal{T}$ (which agrees with the spectrum, in the sense of [1], of the compacts) has two points $\langle k_x \rangle = \sqrt{\mathcal{T}}$ and $\langle k_x, k_y \rangle$. It is the same as the topological space underlying the spectrum of a discrete valuation ring. Both of these ideals are Bousfield classes:

$$\langle k_x \rangle = A(k_y) \quad \text{and} \quad \langle k_x, k_y \rangle = A(k_x).$$

Denoting by $\mathrm{Loc}^{\sqrt{\otimes}}(\mathcal{T})$ the collection of radical localizing $\otimes$-ideals we have

$$A(\mathcal{T}) = \mathrm{Loc}^{\sqrt{\otimes}}(\mathcal{T}) \cup \{\langle 0 \rangle\} = \{\mathcal{T}, \langle k_x \rangle, \langle k_x, k_y \rangle, \langle 0 \rangle\}.$$

There is also a non-radical ideal, namely $\langle k_y \rangle$, which is not a Bousfield class.

It is easily seen that forming the quotient $\mathcal{S} = \mathcal{T}/\sqrt{\mathcal{T}}$ does not change the spectrum and gives a bijection $\mathrm{Loc}^{\sqrt{\otimes}}(\mathcal{T}) \cong \mathrm{Loc}^{\sqrt{\otimes}}(\mathcal{S})$. However, it is no longer true that every radical ideal is a Bousfield class. Indeed, the generic point of the global spectrum of $\mathcal{S}$ is $\langle k_y \rangle$ and this is not a Bousfield class.

References

[1] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168.

[2] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281.

[3] Mark Hovey and John H. Palmieri, *The structure of the Bousfield lattice*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), 1999, pp. 175–196.

[4] S. Iyengar and H. Krause, *The Bousfield lattice of a triangulated category and stratification*, 2011. arXiv:1105.1799.

[5] E. M. Vechtomov, *Annihilator characterizations of Boolean rings and Boolean lattices*, Mat. Zametki **53** (1993), no. 2, 15–24.