ON THE FAILURE OF FUNCTORIAL CONES IN TRIANGULATED CATEGORIES

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1. Introduction

We give a proof that if it is possible to make the choice of cones in an idempotent complete triangulated category $\mathcal{T}$ functorial i.e., to produce a cone functor $\mathcal{T}^{[1]} \longrightarrow \mathcal{T}$, then $\mathcal{T}$ is semisimple abelian. This fact can be found in Verdier's thesis ([1, Proposition 1.2.13]), which I had unfortunately forgotten when I got asked about this and wrote this note. However, we give a "different" proof (really the idea is the same though) and note that Verdier's hypothesis that $\mathcal{T}$ have countable products or coproducts is not necessary; it is sufficient to have split idempotents.

2. Weak limits and colimits

In order to prove the "main result" we need some preliminaries on weak limits and colimits; since I was unable to find a reference for the, presumably well known, result we will use the details are included. Let us begin by recalling the definition of a weak (co)limit. We point out that in this section cocone means a cocone under a diagram (as in standard category theory usage); no categories in this section are necessarily triangulated so this is not some perverse terminology for homotopy fibre. All of our definitions and results will be stated for colimits; of course all of these definitions and results have duals concerning limits (which will also be used in the next section).

Definition 2.1. Let $\mathcal{C}$ be a category, I a small category, and $F\colon I \longrightarrow \mathcal{C}$ a diagram of shape I in $\mathcal{C}$. A *weak colimit* for F is a cocone under the diagram F, say $F \longrightarrow x$, satisfying the existence property of a colimit but not necessarily the universality - given any $F \longrightarrow y$ there exists a map $x \longrightarrow y$ through which the cocone $F \longrightarrow y$ factors.

A *functorial weak colimit* (x, γ) of F consists of a weak colimit $F \longrightarrow x$ and a coherent choice of factorizations $\gamma_{F\longrightarrow y}\colon x \longrightarrow y$ indexed by cocones $F \longrightarrow y$ i.e., one can choose factorizations in such a way that they give a functor from the category of cocones with base F to the slice category under x and we have fixed such a choice.

Lemma 2.2. *Suppose $\mathcal{C}$ is a category with split idempotents. If $\mathcal{C}$ admits a functorial weakly initial object (i, δ) (i.e. a functorial weak colimit of the empty diagram) then $\mathcal{C}$ admits an initial object.*

Proof. Consider the morphism $\delta_i\colon i \longrightarrow i$ which is part of the structure making i weakly functorially initial. First observe that if $\delta_i = 1_i$ then i is actually initial. Indeed, in this case if $c \in \mathcal{C}$ and $\alpha \in \mathcal{C}(i, c)$ then commutativity of the triangle

$$\begin{tikzcd} i \arrow[r, "\delta_i"] \arrow[d, "\delta_c"'] & i \arrow[dl, "\alpha"] \\ c & \end{tikzcd}$$

forces $\alpha = \delta_c$. If δ_i is an isomorphism i is also initial as functoriality forces $\delta_i = 1_i$ via

$$\begin{tikzcd} i \arrow[r, "\delta_i"] \arrow[d, "\delta_i"'] & i \arrow[dl, "\delta_i^{-1}"] \\ i & \end{tikzcd}$$

So we may assume δ_i is not an isomorphism. Then functoriality shows that δ_i is idempotent so, by assumption, it splits and we get a retract $i' \xrightarrow{e} i \xrightarrow{f} i'$. We claim that i' is an initial object. Setting $\delta' = \delta e$ it is clear that the pair (i', δ') is functorially weakly initial. To complete the proof observe that the triangle

$$\begin{tikzcd} i' \arrow[r, "\delta'_i"] \arrow[d, "\delta'_{i'}"'] & i \arrow[dl, "f"] \\ i' & \end{tikzcd}$$

together with $\delta'_i = \delta_i e = efe = e$ yields $\delta'_{i'} = fe = 1_{i'}$. □

Proposition 2.3. *Suppose $\mathcal{C}$ is a category with split idempotents, I is a small category, and $F\colon I \longrightarrow \mathcal{C}$ is a diagram. If $\mathcal{C}$ admits a functorial weak colimit (x, γ) for F then F has a colimit in $\mathcal{C}$. In particular, if $\mathcal{C}$ has functorial weak colimits (of shape I) then it has colimits (of shape I).*

Proof. This is all essentially immediate from the lemma. The existence of a functorial weak colimit gives rise to a functorial weakly initial object in the comma category $(F \downarrow \Delta)$, where $\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^I$ is the diagonal. Since this category inherits the property of having split idempotents from $\mathcal{C}$ we see that it then has an initial object and this is precisely the colimit of F. □

3. This is why we can't have nice things

We now use the abstract nonsense of the first section to prove that the triangulated categories admitting functorial cones carry rather special triangulations.

Proposition 3.1. *Let $\mathcal{T}$ be an idempotent complete triangulated category. If $\mathcal{T}$ admits functorial cones then $\mathcal{T}$ is semisimple abelian.*

Proof. Since the cone of a morphism $X \longrightarrow Y$ in $\mathcal{T}$ is a weak cokernel we see that $\mathcal{T}$ has functorial weak cokernels. In particular, Proposition 2.3 tells us that $\mathcal{T}$ has cokernels. Similarly $\mathcal{T}$ has functorial weak kernels and hence kernels. Since monos and epis split in $\mathcal{T}$ it follows easily that $\mathcal{T}$ is in fact abelian and semisimple. □

In fact the same idea gives the following refined statement.

Proposition 3.2. *Let $\mathcal{T}$ be an idempotent complete triangulated category. If $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a triangle such that completing morphisms of squares to morphisms of triangles as in [TR3] can be done functorially then $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a sum of suspensions of triangles of the form $A \longrightarrow A \longrightarrow 0 \longrightarrow \Sigma A$.*

We now show that even without idempotent completeness of $\mathcal{T}$ one can show that functorial cones force $\mathcal{T}$ to be close to abelian.

Proposition 3.3. *If $\mathcal{T}$ is a triangulated category with functorial cones then $\mathcal{T}$ is a full extension closed subcategory of a semisimple abelian category.*

Proof. Taking idempotent completions is a functor so functorial cones for $\mathcal{T}$ gives functorial cones for the idempotent completion $\tilde{\mathcal{T}}$. Thus $\tilde{\mathcal{T}}$ is semisimple abelian and $\mathcal{T}$ has a full exact embedding into $\tilde{\mathcal{T}}$. □

References

[1] J-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1996), xii+253 pp. (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

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