Complex Analysis: Exercise 11

1. Prove that the function
$$\frac{\pi^2}{\sin^2 \pi z}$$
is analytic everywhere in $\mathbb{C} \setminus \mathbb{Z}$, and the points of $\mathbb{Z}$ are isolated poles of the function.

2. Show that the function
$$g(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$
has removable singularities at all points of $\mathbb{Z}$, so that we can take g to be an entire function $g : \mathbb{C} \to \mathbb{C}$.

3. Show that for $z = x + iy$, we have both
$$\lim_{y \to \pm\infty} \frac{\pi^2}{\sin^2 \pi z} = 0$$
and
$$\lim_{y \to \pm\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = 0.$$

4. Why must we then have $g(z) = 0$, for all z?