Complex Analysis: Exercise 12

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(z) \neq 0$ for all $z \in \mathbb{C}$. Show that there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ with
$$f(z) = e^{g(z)}$$
for all z.

2. Using partial integration, prove that for $\mathrm{Re}(z) > 0$, and $n \in \mathbb{N}$, we have
$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt = \frac{n^z n!}{z(z+1)\cdots(z+n)}.$$

3. For $0 \leq t \leq n$ we have
$$\left(1 - \frac{t}{n}\right)^n \leq e^{-t}.$$
Show that we then have
$$\int_0^\infty e^{-t} t^{z-1}\,dt = \int_0^\infty \left(\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n\right) t^{z-1}\,dt = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt$$
for $\mathrm{Re}(z) > 1$.