Complex Analysis: Exercise 4

1. Show that the function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is an entire function with no zeros. (That is, there exists no $z \in \mathbb{C}$ with $\exp(z) = 0$.) On the other hand, show that for all $z_0 \neq 0$, there exist infinitely many different solutions to the equation

$$\exp(z) = z_0.$$

2. In real analysis, the exponential function is a monomorphism and so the the logarithm function is simply defined to be it's inverse. As we see from the above exercise, that is no longer possible here. On the other hand, we also recall that the logarithm function was the antiderivative of the function $1/x$. Does the function $1/x$ have an antiderivative in the complex plane?

   One way of thinking about this is to choose some $z_0 \neq 0$, and let w_0 be some particular complex number with $\exp(w_0) = z_0$. The most obvious example is $z_0 = 1$ and $w_0 = 0$. Can you find a power series representation of the "branch" of the logarithm function giving this solution? That is, find complex numbers c_n such that the power series

$$f(z) = \sum_{n=1}^{\infty} c_n (z-1)^n$$

   has a positive radius of convergence, and $\exp(f(z)) = z$ for all z within the circle of convergence.

3. Let $f : G \to \mathbb{C}$ be analytic, and assume that

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset G$$

   (where $r > 0$). Let $\gamma(t) = z_0 + re^{2\pi i t}$ be the boundary curve of the disc. Show that

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{(z - z_0)^2}.$$