Complex Analysis: Exercise 6

1. Let X be a topological space, $x_0 \in X$ some element, and consider the set of all continuous closed paths $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. Two such paths are *equivalent* if they are homotopic. The equivalance classes are considered to be the elements of a group, namely the *fundamental group* $\pi_1(X, x_0)$. Multiplication within this group is defined using the following rule. Let γ and ζ be two such closed paths. Then the *product* of the closed paths is the path given by

$$\gamma \cdot \zeta(t) = \begin{cases} \gamma(2t), & t \leq 1/2 \\ \zeta(2t-1), & t \geq 1/2. \end{cases}$$

The product of the two equivalence classes in the fundamental group is then the equivalance class of this product.

   (a) Show that this product operation in $\pi_1(X, x_0)$ is well defined.

   (b) Show that there exists an identity element, and each element has an inverse.

   (c) Finally, show that the associative law holds in $\pi_1(X, x_0)$.

2. Let $G = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. Show that $\pi_1(G, 1) \cong \mathbb{Z}$. That is to say, the fundamental group of G is isomorphic to the integers, considered as a group with the addition operation.

3. Is the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

convergent, for all $z \notin \mathbb{Z}$?