Complex Analysis: Exercise 8

1. Assume that R is a rational function defined throughout $\mathbb{C}$, and $0 < \lambda < 1$ is a real number. Assume that R has a zero of order at least 2 at ∞, and a simple pole (of order 1) at 0. Then let

$$f(z) = z^{\lambda} R(z).$$

For $r, R > 0$, show that

(a)

$$\lim_{r \to 0} \int_{|z|=r} f(z) dz = 0,$$

(b)

$$\lim_{R \to \infty} \int_{|z|=R} f(z) dz = 0.$$

2. Let $a \in \mathbb{R}$ with $a > 1$. What is

$$\int_{|z|=1} \frac{dz}{z^2 + 2az + 1} \quad ?$$

3. Remembering that for $|z| = 1$, we have $z = \cos\theta + i\sin\theta$, for some $\theta \in [0, 2\pi)$, can you show that

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = -i \int_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

where $a > 1$?