KRULL-SCHMIDT CATEGORIES AND PROJECTIVE COVERS

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Abstract. Krull-Schmidt categories are additive categories such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings. We provide a self-contained introduction which is based on the concept of a projective cover.

Contents

1. Introduction

Krull-Schmidt categories are ubiquitous in algebra and geometry; they are additive categories such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings. Such decompositions are essentially unique. Important examples are categories of modules having finite composition length.

The aim of this note is to explain the concept of a Krull-Schmidt category in terms of projective covers. For instance, the uniqeness of direct sum decompositions in Krull-Schmidt categories follows from the uniqueness of projective covers (Theorem 4.2). The exposition is basically self-contained. The results are somewhat classical, but it seems hard to find the material in the literature.

The term 'Krull-Schmidt category' refers to a result known as 'Krull-Remak-Schmidt theorem'. This formulates the existence and uniqueness of the decomposition of a finite length module into indecomposable ones [7, 8, 9]. Atiyah [1] established an analogue for coherent sheaves which is based on a chain condition for objects of an abelian category (Theorem 5.5).

The abstract concept of a Krull-Schmidt category can be found, for example, in expositions of Auslander [2, 3] and Gabriel-Roiter [5]. The basic idea is always to translate properties of an additive category into properties of modules over some appropriate endomorphism ring. Thus we see that an additive category is a Krull-Schmidt category if and only if it has split idempotents and the endomorphism ring of every object is semi-perfect (Corollary 4.4). Essential ingredients of this discussion are the radical of an additive category [6] and the concept of a projective cover [4].

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2. Additive categories and the radical

Products and coproducts. Let $\mathcal{A}$ be a category. A *product* of a family $(X_i)_{i\in I}$ of objects of $\mathcal{A}$ is an object X together with morphisms $\pi_i\colon X\to X_i$ $(i\in I)$ such that for each object A and each family of morphisms $\phi_i\colon A\to X_i$ $(i\in I)$ there exists a unique morphism $\phi\colon A\to X$ with $\phi_i=\pi_i\phi$ for all i. The product solves a 'universal problem' and is therefore unique up to a unique isomorphism; it is denoted by $\prod_{i\in I}X_i$ and is characterized by the fact that the π_i induce a bijection

$$\mathrm{Hom}_{\mathcal{A}}(A,\prod_{i\in I}X_i)\xrightarrow{\sim}\prod_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A,X_i),$$

where the second product is taken in the category of sets.

The *coproduct* $\coprod_{i\in I}X_i$ is the dual notion; it comes with morphisms $\iota_i\colon X_i\to\coprod_{i\in I}X_i$ which induce a bijection

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I}X_i,A)\xrightarrow{\sim}\prod_{i\in I}\mathrm{Hom}_{\mathcal{A}}(X_i,A).$$

Additive categories. A category $\mathcal{A}$ is *additive* if

(1) each morphism set $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is an abelian group and the composition maps
$$\mathrm{Hom}_{\mathcal{A}}(Y,Z)\times\mathrm{Hom}_{\mathcal{A}}(X,Y)\longrightarrow\mathrm{Hom}_{\mathcal{A}}(X,Z)$$
are bilinear,

(2) there is a *zero object* 0, that is, $\mathrm{Hom}_{\mathcal{A}}(X,0)=0=\mathrm{Hom}_{\mathcal{A}}(0,X)$ for every object X, and

(3) every pair of objects X,Y admits a product $X\prod Y$.

Direct sums. Let $\mathcal{A}$ be an additive category. Given a finite number of objects $X_1,\dots,X_r$ of $\mathcal{A}$, a *direct sum*

$$X=X_1\oplus\ldots\oplus X_r$$

is by definition an object X together with morphisms $\iota_i\colon X_i\to X$ and $\pi_i\colon X\to X_i$ for $1\le i\le r$ such that $\sum_{i=1}^r\iota_i\pi_i=\mathrm{id}_X$ and $\pi_i\iota_i=\mathrm{id}_{X_i}$ for all i.

Lemma 2.1. *The morphisms ι_i and π_i induce isomorphisms*

$$\coprod_{i=1}^r X_i\cong\bigoplus_{i=1}^r X_i\cong\prod_{i=1}^r X_i.$$

Proof. A morphism $X\to Y$ in $\mathcal{A}$ is an isomorphism if it induces for each object A an isomorphism $\mathrm{Hom}_{\mathcal{A}}(A,X)\to\mathrm{Hom}_{\mathcal{A}}(A,Y)$ of abelian groups. The functor $\mathrm{Hom}_{\mathcal{A}}(A,-)$ sends the direct sum $\bigoplus_i X_i$ in $\mathcal{A}$ to a direct sum $\bigoplus_i\mathrm{Hom}_{\mathcal{A}}(A,X_i)$ of abelian groups. It is a standard fact that finite direct sums and products of abelian groups are isomorphic. Thus the following composite is in fact an isomorphism.

$$\bigoplus_i\mathrm{Hom}_{\mathcal{A}}(A,X_i)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{A}}(A,\bigoplus_i X_i)\to\mathrm{Hom}_{\mathcal{A}}(A,\prod_i X_i)\xrightarrow{\sim}\prod_i\mathrm{Hom}_{\mathcal{A}}(A,X_i).$$

This establishes the isomorphism $\bigoplus_i X_i\cong\prod_i X_i$ and the other isomorphism $\coprod_i X_i\cong\bigoplus_i X_i$ follows by symmetry. ☐

Lemma 2.1 implies that a direct sum of $X_1,\dots,X_r$ is unique up to a unique isomorphism. Thus one may speak of *the* direct sum and the notation $X_1\oplus\ldots\oplus X_r$ is well-defined. We write $X^r=X\oplus\ldots\oplus X$ for the direct sum of r copies of an object X.

Let $X = X_1 \oplus \ldots \oplus X_r$ and $Y = Y_1 \oplus \ldots \oplus Y_s$ be two direct sums. Then one has

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(X_i, Y_j)$$

and therefore each morphism $\phi\colon X \to Y$ can be written uniquely as a matrix $\phi = (\phi_{ij})$ with entries $\phi_{ij} = \pi_j \phi \iota_i$ in $\mathrm{Hom}_{\mathcal{A}}(X_i, Y_j)$ for all pairs i, j.

A non-zero object X is *indecomposable* if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

An additive category has *split idempotents* if every idempotent endomorphism $\phi = \phi^2$ of an object X splits, that is, there exists a factorisation $X \xrightarrow{\pi} Y \xrightarrow{\iota} X$ of ϕ with $\pi\iota = \mathrm{id}_Y$.

Given an object X in an additive category, we denote by $\operatorname{add} X$ the full subcategory consisting of all finite direct sums of copies of X and their direct summands. This is the smallest additive subcategory which contains X and is closed under taking direct summands.

Abelian categories. An additive category $\mathcal{A}$ is *abelian*, if every morphism $\phi\colon X \to Y$ has a kernel and a cokernel, and if the canonical factorisation

$$\begin{tikzcd}
\operatorname{Ker}\phi \arrow[r, "\phi'"] & X \arrow[r, "\phi"] \arrow[d] & Y \arrow[r, "\phi''"] & \operatorname{Coker}\phi \\
& \operatorname{Coker}\phi' \arrow[r, "\bar{\phi}"] & \operatorname{Ker}\phi'' \arrow[u] &
\end{tikzcd}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Example 2.2. Let Λ be an associative ring.

(1) The category $\operatorname{Mod}\Lambda$ of right Λ-modules is an abelian category.

(2) The category $\operatorname{proj}\Lambda$ of finitely generated projective Λ-modules is an additive category. This category has split idempotents and equals the subcategory $\operatorname{add}\Lambda$ of $\operatorname{Mod}\Lambda$ which is given by Λ viewed as a Λ-module.

Projectivisation. Every object of an additive category can be turned into a finitely generated projective module over its endomorphism ring.

Proposition 2.3. *Let $\mathcal{A}$ be an additive category and X an object with $\Gamma = \mathrm{End}_{\mathcal{A}}(X)$. The functor $\mathrm{Hom}_{\mathcal{A}}(X,-)\colon \mathcal{A} \to \operatorname{Mod}\Gamma$ induces a fully faithful functor $\operatorname{add} X \to \operatorname{proj}\Gamma$. This functor is an equivalence if $\mathcal{A}$ has split idempotents.*

Proof. We need to show that $F = \mathrm{Hom}_{\mathcal{A}}(X,-)$ induces a bijection

$$\mathrm{Hom}_{\mathcal{A}}(X', X'') \longrightarrow \mathrm{Hom}_{\Gamma}(FX', FX'')$$

for all X', X'' in $\operatorname{add} X$. Clearly, the map is a bijection for $X' = X = X''$ since $FX = \Gamma$. From this the general case follows because F is additive and $\operatorname{proj}\Gamma = \operatorname{add}\Gamma$. Every object in $\operatorname{proj}\Gamma$ is a direct summand of Γ^n for some n and therefore isomorphic to one in the image of F if $\mathcal{A}$ has split idempotents. In that case F induces an equivalence between $\operatorname{add} X$ and $\operatorname{proj}\Gamma$. $\square$

Remark 2.4. Every additive category $\mathcal{A}$ admits an *idempotent completion* $F\colon \mathcal{A} \to \bar{\mathcal{A}}$, that is, $\bar{\mathcal{A}}$ is an additive category with split idempotents and the functor F is fully faithful, additive, and each object in $\bar{\mathcal{A}}$ is a direct summand of an object in the image of F. For instance, if $\mathcal{A} = \operatorname{add} X$ for some object X with $\Gamma = \mathrm{End}_{\mathcal{A}}(X)$, then one takes $\bar{\mathcal{A}} = \operatorname{proj}\Gamma$ and $F = \mathrm{Hom}_{\mathcal{A}}(X,-)$.

Subobjects. Let $\mathcal{A}$ be an abelian category. We say that two monomorphisms $X_1 \to X$ and $X_2 \to X$ are *equivalent*, if there exists an isomorphism $X_1 \xrightarrow{\sim} X_2$ making the following diagram commutative.

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & X & \end{tikzcd}$$

An equivalence class of monomorphisms into X is called a *subobject* of X. Given subobjects $X_1 \to X$ and $X_2 \to X$, we write $X_1 \subseteq X_2$ if there is a morphism $X_1 \to X_2$ making the above diagram commutative.

An object $X \neq 0$ is *simple* if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

Given a family of subobjects $(X_i)_{i \in I}$ of an object X, let $\sum_{i \in I} X_i$ denote the smallest subobject of X containing all X_i, provided such an object exists. If the coproduct $\coprod_{i \in I} X_i$ exists in $\mathcal{A}$, then $\sum_{i \in I} X_i$ equals the image of the canonical morphism $\coprod_{i \in I} X_i \to X$. The family of subobjects $(X_i)_{i \in I}$ is *directed* if for each pair $i, j \in I$, there exists $k \in I$ with $X_i, X_j \subseteq X_k$.

An object X is *finitely generated* if $X = \sum_{i \in I} X_i$ for some directed set of subobjects $X_i \subseteq X$ implies $X = X_{i_0}$ for some index $i_0 \in I$.

Lemma 2.5. *Let X be a finitely generated object. Suppose that the subobjects of X form a set and that $\sum_{i \in I} X_i$ exists for every family of subobjects $(X_i)_{i \in I}$. Then every proper subobject of X is contained in a maximal subobject.*

Proof. Apply Zorn's lemma. □

Example 2.6. A Λ-module X is finitely generated if and only if there exist elements $x_1, \ldots, x_n$ in X such that $X = \sum_i x_i \Lambda$.

The Jacobson radical. Let X be an object in an abelian category. The *radical* of X is the intersection of all its maximal subobjects and is denoted by $\operatorname{rad} X$. Note that $\phi(\operatorname{rad} X) \subseteq \operatorname{rad} Y$ for every morphism $\phi \colon X \to Y$. Thus the assignment $X \mapsto \operatorname{rad} X$ defines a subfunctor of the identity functor.

For a ring Λ, the radical of the Λ-module Λ is called *Jacobson radical* and will be denoted by $J(\Lambda)$. The following lemma implies that $J(\Lambda)$ is a two-sided ideal.

Lemma 2.7 (Nakayama). *Let X be a Λ-module. Then $XJ(\Lambda) \subseteq \operatorname{rad} X$. In particular, $XJ(\Lambda) = X$ implies $X = 0$ provided that X is finitely generated.*

Proof. For any $x \in X$, left multiplication with x induces a morphism $\Lambda \to X$, and therefore $x(\operatorname{rad} \Lambda) \subseteq \operatorname{rad} X$.

If X is finitely generated, then every proper submodule is contained in a maximal submodule. Thus $\operatorname{rad} X = X$ implies $X = 0$. □

The next lemma gives a more explicit description of the Jacobson radical. In particular, one sees that it is a left-right symmetric concept.

Lemma 2.8. *Let Λ be a ring. Then*

$$\begin{aligned} J(\Lambda) &= \{x \in \Lambda \mid 1 - xy \text{ has a right inverse for all } y \in \Lambda\} \\ &= \{x \in \Lambda \mid 1 - y'xy \text{ is invertible for all } y, y' \in \Lambda\}. \end{aligned}$$

In particular, $J(\Lambda^{\mathrm{op}}) = J(\Lambda)$.

Proof. We have $x \in J(\Lambda)$ if and only if $\mathfrak{m} + x\Lambda \neq \Lambda$ for every maximal right ideal $\mathfrak{m}$, and this is equivalent to $1 - xy \notin \mathfrak{m}$ for every $y \in \Lambda$ and maximal $\mathfrak{m}$, that is, $1 - xy$ has a right inverse. This establishes the first equality.

For the second equality, it remains to show that $x \in J(\Lambda)$ implies $1-x$ is invertible. We know there exists z such that $(1-x)z = 1$. Thus $1-z = -xz \in J(\Lambda)$, so there exists z' such that $(1-(1-z))z' = 1$, that is, $zz' = 1$. Hence z is invertible, and so is then also $1-x$. □

The radical of an additive category. Let $\mathcal{A}$ be an additive category. A *two-sided ideal* $\mathfrak{I}$ of $\mathcal{A}$ consists of subgroups $\mathfrak{I}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{A}}(X,Y)$ for each pair of objects $X, Y \in \mathcal{A}$ such that for every sequence $X' \xrightarrow{\sigma} X \xrightarrow{\phi} Y \xrightarrow{\tau} Y'$ of morphisms in $\mathcal{A}$ with $\phi \in \mathfrak{I}(X,Y)$ the composite $\tau\phi\sigma$ belongs to $\mathfrak{I}(X',Y')$. Note that a morphism $(\phi_{ij})\colon \bigoplus_i X_i \to \bigoplus_j Y_j$ belongs to an ideal $\mathfrak{I}$ if and only if $\phi_{ij} \in \mathfrak{I}$ for all i, j.

Given a pair X, Y of objects of $\mathcal{A}$, we define the *radical*

$$\operatorname{Rad}_{\mathcal{A}}(X,Y) := \{\phi \in \operatorname{Hom}_{\mathcal{A}}(X,Y) \mid \phi\psi \in J(\operatorname{End}_{\mathcal{A}}(Y)) \text{ for all } \psi \in \operatorname{Hom}_{\mathcal{A}}(Y,X)\}.$$

It follows from Lemma 2.8 that $\phi \in \operatorname{Hom}_{\mathcal{A}}(X,Y)$ belongs to the radical if and only if $\mathrm{id}_Y - \phi\psi$ has a right inverse for every $\psi \in \operatorname{Hom}_{\mathcal{A}}(Y,X)$.

Proposition 2.9. *The radical* $\operatorname{Rad}_{\mathcal{A}}$ *is the unique two-sided ideal of* $\mathcal{A}$ *such that* $\operatorname{Rad}_{\mathcal{A}}(X,X) = J(\operatorname{End}_{\mathcal{A}}(X))$ *for every object* $X \in \mathcal{A}$.

Proof. Each set $\operatorname{Rad}_{\mathcal{A}}(X,Y)$ is a subgroup of $\operatorname{Hom}_{\mathcal{A}}(X,Y)$ since $J(\operatorname{End}_{\mathcal{A}}(Y))$ is a subgroup of $\operatorname{End}_{\mathcal{A}}(Y)$. Now fix a sequence $X' \xrightarrow{\sigma} X \xrightarrow{\phi} Y \xrightarrow{\tau} Y'$ of morphisms in $\mathcal{A}$ with $\phi \in \operatorname{Rad}_{\mathcal{A}}(X,Y)$. Clearly, $\phi\sigma \in \operatorname{Rad}_{\mathcal{A}}(X',Y)$ and it remains to show that $\tau\phi \in \operatorname{Rad}_{\mathcal{A}}(X,Y')$. We use the description of the Jacobson radical in Lemma 2.8. Choose $\psi \in \operatorname{Hom}_{\mathcal{A}}(Y',X)$. Then $\mathrm{id}_Y - \phi\psi\tau$ has a right inverse, say $\alpha \in \operatorname{End}_{\mathcal{A}}(Y)$, since $\phi \in \operatorname{Rad}_{\mathcal{A}}(X,Y)$. A simple calculation shows that $(\mathrm{id}_{Y'} - \tau\phi\psi)(\mathrm{id}_{Y'} + \tau\alpha\phi\psi) = \mathrm{id}_{Y'}$. It follows that $\tau\phi$ belongs to $\operatorname{Rad}_{\mathcal{A}}(X,Y')$. Thus $\operatorname{Rad}_{\mathcal{A}}$ is a two-sided ideal of $\mathcal{A}$.

It is clear from the definition that $\operatorname{Rad}_{\mathcal{A}}(X,X) = J(\operatorname{End}_{\mathcal{A}}(X))$ for every $X \in \mathcal{A}$. Any two-sided ideal $\mathfrak{I}$ of $\mathcal{A}$ is determined by the collection of subgroups $\mathfrak{I}(X,X)$, where X runs through all objects of $\mathcal{A}$. In fact, a morphism $\phi \in \operatorname{Hom}_{\mathcal{A}}(X,Y)$ belongs to $\mathfrak{I}(X,Y)$ if and only if $\left[\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right]$ belongs to $\mathfrak{I}(X \oplus Y, X \oplus Y)$. □

The following description of the radical $\operatorname{Rad}_{\mathcal{A}}$ is a consequence; it is symmetric and shows that $\operatorname{Rad}_{\mathcal{A}^{\mathrm{op}}} = \operatorname{Rad}_{\mathcal{A}}$.

Corollary 2.10. *Let* X, Y *be a pair of objects of an additive category* $\mathcal{A}$. *Then the following are equivalent for a morphism* $\phi\colon X \to Y$.

(1) $\phi \in \operatorname{Rad}_{\mathcal{A}}(X,Y)$.
(2) $\mathrm{id}_Y - \phi\psi$ *has a right inverse for all morphisms* $Y \xrightarrow{\psi} X$.
(3) $\tau\phi\sigma \in J(\operatorname{End}_{\mathcal{A}}(Z))$ *for all morphisms* $Y \xrightarrow{\tau} Z \xrightarrow{\sigma} X$.
(4) $\mathrm{id}_Z - \tau\phi\sigma$ *is invertible for all morphisms* $Y \xrightarrow{\tau} Z \xrightarrow{\sigma} X$. □

3. Projective covers

Essential epimorphisms. Let $\mathcal{A}$ be an abelian category. An epimorphism $\phi\colon X \to Y$ is *essential* if any morphism $\alpha\colon X' \to X$ is an epimorphism provided that the composite $\phi\alpha$ is an epimorphism. This condition can be rephrased as follows: If $U \subseteq X$ is a subobject with $U + \operatorname{Ker}\phi = X$, then $U = X$. We collect some basic facts.

Lemma 3.1. *Let* $\phi\colon X \to Y$ *and* $\psi\colon Y \to Z$ *be epimorphisms. Then* $\psi\phi$ *is essential if and only if both* ϕ *and* ψ *are essential.* □

Lemma 3.2. *Let $X_i \to Y_i$ $(i = 1, \dots, n)$ be essential epimorphisms. Then $\bigoplus_i X_i \to \bigoplus_i Y_i$ is an essential epimorphism.*

Proof. It is sufficient to pove the case $n = 2$. In that case write $\bigoplus_i X_i \to \bigoplus_i Y_i$ as composite $X_1 \oplus X_2 \to X_1 \oplus Y_2 \to Y_1 \oplus Y_2$. It is straightforward to check that both morphisms are essential. Thus the composite is essential. □

Lemma 3.3. *Let $\phi\colon X \to Y$ be an epimorphism and $U = \operatorname{Ker} \phi$.*

(1) *If ϕ is essential, then $U \subseteq \operatorname{rad} X$.*
(2) *If $U \subseteq \operatorname{rad} X$ and X is finitely generated, then ϕ is essential.*

Proof. (1) Suppose that ϕ is essential and let $V \subseteq X$ be a maximal subobject not containing U. Then $U + V = X$ and therefore $V = X$. This is a contradiction and therefore U is contained in every maximal subobject. This implies $U \subseteq \operatorname{rad} X$.

(2) Suppose that $U \subseteq \operatorname{rad} X$ and let $V \subseteq X$ be a subobject with $U + V = X$. If $V \neq X$, then there is a maximal subobject $V' \subseteq X$ containing V since X is finitely generated; see Lemma 2.5. Thus $X = U + V \subseteq V'$. This is a contradiction and therefore $V = X$. It follows that ϕ is essential. □

Projective covers. Let $\mathcal{A}$ be an abelian category. An epimorphism $\phi\colon P \to X$ is called a *projective cover* of X if P is projective and ϕ is essential.

Lemma 3.4. *Let P be a projective object. Then the following are equivalent for an epimorphism $\phi\colon P \to X$.*

(1) *The morphism ϕ is a projective cover of X.*
(2) *Every endomorphism $\alpha\colon P \to P$ satisfying $\phi\alpha = \phi$ is an isomorphism.*

Proof. (1) $\Rightarrow$ (2): Let $\alpha\colon P \to P$ be an endomorphism satisfying $\phi\alpha = \phi$. Then α is an epimorphism since ϕ is essential. Thus there exists $\alpha'\colon P \to P$ satisfying $\alpha\alpha' = \mathrm{id}_P$ since P is projective. It follows that $\phi\alpha' = \phi$ and therefore α' is an epimorphism. On the other hand, α' is a monomorphism. Thus α' and α are isomorphisms.

(2) $\Rightarrow$ (1): Let $\alpha\colon P' \to P$ be a morphism such that $\phi\alpha$ is an epimorphism. Then ϕ factors through $\phi\alpha$ via a morphism $\alpha'\colon P \to P'$ since P is projective. The composite $\alpha\alpha'$ is an isomorphism and therefore α is an epimorphism. Thus ϕ is essential. □

Corollary 3.5. *Let $\phi\colon P \to X$ and $\phi'\colon P' \to X$ be projective covers of an object X. Then there is an isomorphism $\alpha\colon P \to P'$ such that $\phi = \phi'\alpha$.*

A ring is called *local* if the sum of two non-units is again a non-unit.

Lemma 3.6. *Let $\phi\colon P \to S$ be an epimorphism such that P is projective and S is simple. Then the following are equivalent.*

(1) *The morphism ϕ is a projective cover of S.*
(2) *The object P has a maximal subobject that contains every proper subobject of P.*
(3) *The endomorphism ring of P is local.*

Proof. (1) $\Rightarrow$ (2): Let $U \subseteq P$ be a subobject and suppose $U \not\subseteq \operatorname{Ker} \phi$. Then $U + \operatorname{Ker} \phi = P$, and therefore $U = P$ since ϕ is essential. Thus $\operatorname{Ker} \phi$ contains every proper subobject of P.

(2) $\Rightarrow$ (3): First observe that P is an indecomposable object. It follows that every endomorphism of P is invertible if and only if it is an epimorphism. Given two non-units α, β in $\operatorname{End}_{\mathcal{A}}(P)$, we have therefore $\operatorname{Im}(\alpha+\beta) \subseteq \operatorname{Im} \alpha + \operatorname{Im} \beta \subseteq \operatorname{rad} P$.

Here we use that $\operatorname{rad} P$ contains every proper subobject of P. Thus $\alpha+\beta$ is a non-unit and $\operatorname{End}_{\mathcal{A}}(P)$ is local.

(3) $\Rightarrow$ (1): Consider the $\operatorname{End}_{\mathcal{A}}(P)$-submodule H of $\operatorname{Hom}_{\mathcal{A}}(P,S)$ which is generated by ϕ. Suppose $\phi=\phi\alpha$ for some α in $\operatorname{End}_{\mathcal{A}}(P)$. If α belongs to the Jacobson radical, then $H=HJ(\operatorname{End}_{\mathcal{A}}(P))$, which is not possible by Lemma 2.7. Thus α is an isomorphism since $\operatorname{End}_{\mathcal{A}}(P)$ is local. It follows from Lemma 3.4 that ϕ is a projective cover. $\square$

Maximal subobjects of projectives. Let $\mathcal{A}$ be an abelian category. We need to assume that for each object X the subobjects of X form a set and that $\sum_{i\in I}X_i$ exists for each family of subobjects $(X_i)_{i\in I}$. Given a subobject $U\subseteq X$, we set

$$\operatorname{End}_{\mathcal{A}}(U|X):=\{\phi\in\operatorname{End}_{\mathcal{A}}(X)\mid \operatorname{Im}\phi\subseteq U\}.$$

Proposition 3.7. *Let $\mathcal{A}$ be an abelian category and X a finitely generated projective object. The maps*

$$X\supseteq U\mapsto \operatorname{End}_{\mathcal{A}}(U|X)\quad \textit{and}\quad \operatorname{End}_{\mathcal{A}}(X)\supseteq\mathfrak{a}\mapsto\sum_{\operatorname{End}_{\mathcal{A}}(V|X)\subseteq\mathfrak{a}}V$$

induces mutually inverse bijections between the maximal subobjects of X and the maximal right ideals of $\operatorname{End}_{\mathcal{A}}(X)$.

Proof. A subobject $U\subseteq X$ induces an exact sequence

$$0\to\operatorname{Hom}_{\mathcal{A}}(X,U)\to\operatorname{Hom}_{\mathcal{A}}(X,X)\to\operatorname{Hom}_{\mathcal{A}}(X,X/U)\to 0.$$

If $U\subseteq X$ is maximal, then $\operatorname{Hom}_{\mathcal{A}}(X,X/U)$ is a simple $\operatorname{End}_{\mathcal{A}}(X)$-module and therefore $\operatorname{End}_{\mathcal{A}}(U|X)$ is a maximal right ideal.

Now fix a maximal right ideal $\mathfrak{m}$ of $\operatorname{End}_{\mathcal{A}}(X)$ and let $U=\sum_{V\in\mathcal{V}}V$ where $\mathcal{V}$ denotes the set of subobjects $V\subseteq X$ with $\operatorname{End}_{\mathcal{A}}(V|X)\subseteq\mathfrak{m}$. First notice that $\mathfrak{m}\subseteq\operatorname{End}_{\mathcal{A}}(U|X)$ since X is projective. Next observe that $\mathcal{V}$ is directed since $V_1,V_2\in\mathcal{V}$ implies $V_1+V_2\in\mathcal{V}$. Thus U is a proper subobject of X since X is finitely generated. In particular, $\operatorname{End}_{\mathcal{A}}(U|X)=\mathfrak{m}$. If $W\subseteq X$ is a subobject properly containing U, then $\operatorname{End}_{\mathcal{A}}(W|X)$ properly contains $\mathfrak{m}$ and equals therefore $\operatorname{End}_{\mathcal{A}}(X)$. Thus $W=X$. It follows that U is maximal. $\square$

Corollary 3.8. *Let $\phi\colon X\to Y$ be a morphism and suppose Y is finitely generated projective. Then $\operatorname{Im}\phi\subseteq\operatorname{rad}Y$ if and only if $\phi\in\operatorname{Rad}_{\mathcal{A}}(X,Y)$.*

Proof. We apply Proposition 3.7. For every maximal subobject $U\subseteq Y$, we have $\operatorname{Im}\phi\subseteq U$ if and only if

$$\operatorname{Im}\operatorname{Hom}_{\mathcal{A}}(Y,\phi)=\operatorname{End}_{\mathcal{A}}(\operatorname{Im}\phi|X)\subseteq\operatorname{End}_{\mathcal{A}}(U|Y).$$

Thus $\operatorname{Im}\phi\subseteq\operatorname{rad}Y$ if and only if $\operatorname{Im}\operatorname{Hom}_{\mathcal{A}}(Y,\phi)\subseteq J(\operatorname{End}_{\mathcal{A}}(Y))$. It follows that ϕ belongs to $\operatorname{Rad}_{\mathcal{A}}(X,Y)$. $\square$

Remark 3.9. The assumption on Y to be projective is necessary in Corollary 3.8. Take for instance over $\Lambda=k[x,y]/(x^2,y^2)$ (k any field) the module $Y=\operatorname{rad}\Lambda$ and let $\phi\colon X\to Y$ be the inclusion of a maximal submodule X. Then $\phi\in\operatorname{Rad}_{\Lambda}(X,Y)$ but $\operatorname{Im}\phi=X\not\subseteq\operatorname{rad}Y$.

Projective presentations. Let $\mathcal{A}$ be an abelian category. An exact sequence $P_1\to P_0\to X\to 0$ is called a *projective presentation* of X if P_0 and P_1 are projective objects.

Proposition 3.10. *Let $P_1\xrightarrow{\phi}P_0\xrightarrow{\psi}X\to 0$ be a projective presentation. Then ψ is a projective cover of X if and only if ϕ belongs to $\operatorname{Rad}_{\mathcal{A}}(P_1,P_0)$.*

Proof. Let $P = P_0 \oplus P_1$ and $\Gamma = \operatorname{End}_{\mathcal{A}}(P)$. Denote by $\mathcal{C}$ the smallest full additive subcategory of $\mathcal{A}$ containing P and closed under taking cokernels. Using Proposition 2.3, it is not hard to verify that $F = \operatorname{Hom}_{\mathcal{A}}(P, -) \colon \mathcal{A} \to \operatorname{Mod}\Gamma$ induces an equivalence $\mathcal{C} \xrightarrow{\sim} \operatorname{mod}\Gamma$, where $\operatorname{mod}\Gamma$ denotes the category of finitely presented Γ-modules.

It follows from Lemma 3.4 that ψ is a projective cover of X if and only if $F\psi$ is a projective cover of FX. The module FP_0 is finitely generated and therefore $F\psi$ is a projective cover if and only if $\operatorname{Ker} F\psi \subseteq \operatorname{rad} FP_0$, by Lemma 3.3. Finally, Corollary 3.8 implies that $\operatorname{Ker} F\psi \subseteq \operatorname{rad} FP_0$ if and only if $F\phi$ belongs to $\operatorname{Rad}_\Gamma(FP_0, FP_1)$. It remains to note that F induces a bijection $\operatorname{Rad}_{\mathcal{A}}(P_0, P_1) \cong \operatorname{Rad}_\Gamma(FP_0, FP_1)$. $\square$

4. Krull-Schmidt categories

Krull-Schmidt categories. An additive category is called *Krull-Schmidt category* if every object decomposes into a finite direct sum of objects having local endomorphism rings.

Proposition 4.1. *For a ring Λ the following are equivalent.*

1. *The category of finitely generated projective Λ-modules is a Krull-Schmidt category.*
2. *The module Λ admits a decomposition $\Lambda = P_1 \oplus \ldots \oplus P_r$ such that each P_i has a local endomorphism ring.*
3. *Every simple Λ-module admits a projective cover.*
4. *Every finitely generated Λ-module admits a projective cover.*

A ring is *semi-perfect* if it satisfies the equivalent conditions in the preceding proposition.

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): Let S be a simple Λ-module. Then we have a non-zero morphism $\Lambda \to S$ and therefore a non-zero morphism $\phi \colon P \to S$ for some indecomposable direct summand P of Λ. The morphism ϕ is a projective cover by Lemma 3.6, because $\operatorname{End}_\Lambda(P)$ is local.

(3) $\Rightarrow$ (1): Let P be a finitely generated projective Λ-module. We claim that $P/\operatorname{rad} P$ is semi-simple. To prove this, let $P'/\operatorname{rad} P \subseteq P/\operatorname{rad} P$ be the sum of all simple submodules. If $P' \neq P$, there is a maximal submodule $U \subseteq P$ containing P', and the simple module P/U admits a projective cover $\pi \colon Q \to P/U$. The morphism $P \to P/U$ factors through $Q \to P/U$ via a morphism $\phi \colon P \to Q$. Analogously, there is a morphism $\psi \colon Q \to P$, and the composite $\phi\psi$ is an isomorphism since π is a projective cover, by Lemma 3.4. Observe that $\operatorname{Ker}\pi = \operatorname{rad} Q$, by Lemma 3.6. Thus $P/U \cong Q/\operatorname{rad} Q$, and therefore ψ induces a right inverse for the canonical morphism $P/\operatorname{rad} P \to P/U$. This contradicts the property of $P'/\operatorname{rad} P$ to contain all simple submodules of $P/\operatorname{rad} P$. It follows that $P/\operatorname{rad} P$ is semi-simple. Let $P/\operatorname{rad} P = \bigoplus_i S_i$ be a decomposition into finitely many simple modules and choose a projective cover $P_i \to S_i$ for each i. Then $P \cong \bigoplus_i P_i$, since $P \to P/\operatorname{rad} P$ and $\bigoplus_i P_i \to \bigoplus_i S_i$ are both projective covers. It remains to observe that each P_i is indecomposable with a local endomorphism ring, by Lemma 3.6.

(1) & (3) $\Rightarrow$ (4): The assumption implies that every finite sum of simple Λ-modules admits a projective cover; see Lemma 3.2. Now let X be a finitely generated Λ-module and choose an epimorphism $\phi \colon P \to X$ with P finitely generated projective. Let $P = \bigoplus_{i=1}^n P_i$ be a decomposition into indecomposable modules.

Then

$$P/\operatorname{rad} P = \bigoplus_{i=1}^{n} P_i/\operatorname{rad} P_i$$

is a finite sum of simple Λ-modules by Lemma 3.6 since each P_i has a local endomorphism ring. The epimorphism ϕ induces an epimorphism $P/\operatorname{rad} P \to X/\operatorname{rad} X$ and therefore $X/\operatorname{rad} X$ decomposes into finitely many simple modules. There exists a projective cover $Q \to X/\operatorname{rad} X$ and this factors through the canonical morphism $\pi\colon X \to X/\operatorname{rad} X$ via a morphism $\psi\colon Q \to X$. The morphism ψ is an epimorphism because π is essential by Lemma 3.3, and Lemma 3.1 implies that ψ is essential.

(4) ⇒ (3): Clear. □

Direct sum decompositions. The uniqueness of direct sum decompositions in Krull-Schmidt categories can be derived from the existence and uniqueness of projective covers over semi-perfect rings.

Theorem 4.2. *Let X be an object of an additive category and suppose there are two decompositions*

$$X_1 \oplus \ldots \oplus X_r = X = Y_1 \oplus \ldots \oplus Y_s$$

into objects with local endomorphism rings. Then $r = s$ and and there exists a permutation π such that $X_i \cong Y_{\pi(i)}$ for $1 \leq i \leq r$.

Proof. Let $\mathcal{A} = \operatorname{add} X$ and identify $\mathcal{A}$ via $\operatorname{Hom}_{\mathcal{A}}(X, -)$ with a full subcategory of the category of finitely generated projective modules over $\operatorname{End}_{\mathcal{A}}(X)$; see Proposition 2.3. Thus we may assume that X is a finitely generated projective module over a semi-perfect ring.

It follows from Lemma 3.6 that for every index i the radical $\operatorname{rad} X_i$ is a maximal submodule of X_i and that the canonical morphism $X_i \to X_i/\operatorname{rad} X_i$ is a projective cover. Thus $X_i \cong Y_j$ if and only if $X_i/\operatorname{rad} X_i \cong Y_j/\operatorname{rad} Y_j$ for every pair i, j, by Corollary 3.5. We have

$$(X_1/\operatorname{rad} X_1) \oplus \ldots \oplus (X_r/\operatorname{rad} X_r) = X/\operatorname{rad} X = (Y_1/\operatorname{rad} Y_1) \oplus \ldots \oplus (Y_s/\operatorname{rad} Y_s)$$

and the assertion now follows from the uniqueness of the decomposition of a semisimple module into simple modules (which is easily proved by induction on the number of summands). □

Corollary 4.3. *Let X be an object of a Krull-Schmidt category and suppose there are two decompositions*

$$X_1 \oplus \ldots \oplus X_n = X = X' \oplus X''$$

such that each X_i is indecomposable. Then there exists an integer $t \leq n$ such that $X = X_1 \oplus \ldots \oplus X_t \oplus X'$ after reindexing the X_i.

Proof. Let $X' = Y_1 \oplus \ldots \oplus Y_s$ and $X'' = Z_1 \oplus \ldots \oplus Z_t$ be decompositions into indecomposable objects. It follows from the uniqueness of these decompositions that $n = s + t$ and that $X'' \cong X_1 \oplus \ldots \oplus X_t$ after some reindexing of the X_i. Composing the decomposition $X = X' \oplus X''$ with that isomorphism yields the assertion. □

Corollary 4.4. *An additive category is a Krull-Schmidt category if and only if it has split idempotents and the endomorphism ring of every object is semi-perfect.*

Proof. The assertion follows from Proposition 4.1 once we know that a Krull-Schmidt category has split idempotents. But this is clear since there is an equivalence $\operatorname{add} X \xrightarrow{\sim} \operatorname{proj} \Gamma$ for $\Gamma = \operatorname{End}_{\mathcal{A}}(X)$, thanks to Proposition 2.3 and Corollary 4.3. □

Let $\mathcal{A}$ be a Krull-Schmidt category and let $X = X_1 \oplus \ldots \oplus X_r$ and $Y = Y_1 \oplus \ldots \oplus Y_s$ be decompositions of two objects X, Y into indecomposable objects. Then we have

$$\operatorname{Rad}_{\mathcal{A}}(X, Y) = \bigoplus_{i,j} \operatorname{Rad}_{\mathcal{A}}(X_i, Y_j)$$

and $\operatorname{Rad}_{\mathcal{A}}(X_i, Y_j)$ equals the set of non-invertible morphisms $X_i \to Y_j$ for each pair i, j.

Example 4.5. The category of finitely generated torsion-free abelian groups admits unique decompositions into indecomposable objects. However, the unique indecomposable object $\mathbb{Z}$ does not have a local endomorphism ring.

5. Chain conditions

The bi-chain condition. A *bi-chain* in a category is a sequence of morphisms $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ such that α_n is an epimorphism and β_n is a monomorphism for all integers $n \geq 0$. The object X satisfies the *bi-chain condition* if for every bi-chain $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ with $X = X_0$ there exists an integer n_0 such that a_n and β_n are invertible for all $n \geq n_0$.

Finite length objects. An object X of an abelian category has *finite length* if there exists a finite chain of subobjects

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X$$

such that each quotient X_i/X_{i-1} is a simple object. Note that X has finite length if and only if X is both *artinian* (i.e. it satisfies the descending chain condition on subobjects) and *noetherian* (i.e. it satisfies the ascending chain condition on subobjects).

Lemma 5.1. *An object of finite length satisfies the bi-chain condition.*

Proof. Let X be an object of finite length and $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ a bi-chain with $X = X_0$. Then the subobjects $\operatorname{Ker}(\alpha_n \ldots \alpha_1 \alpha_0) \subseteq X$ yield an ascending chain and the subobjects $\operatorname{Im}(\beta_0 \beta_1 \ldots \beta_n) \subseteq X$ yield a descending chain. If these chains terminate, then α_n and β_n are invertible for large enough n. □

An additive category $\mathcal{A}$ is *Hom-finite* if there exists a commutative ring k such that $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is a k-module of finite length for all objects X, Y and the composition maps are k-bilinear.

Lemma 5.2. *An object of a Hom-finite abelian category satisfies the bi-chain condition.*

Proof. Let X be an object of a Hom-finite abelian category $\mathcal{A}$ and $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ a bi-chain with $X = X_0$. Each pair α_n, β_n induces a monomorphism $\operatorname{Hom}_{\mathcal{A}}(X_{n+1}, X_{n+1}) \to \operatorname{Hom}_{\mathcal{A}}(X_n, X_n)$. If this map is bijective, then α_n is a monomorphism and β_n is an epimorphism. In an abelian category, any morphism is invertible if it is both a monomorphism and an epimorphism. Thus the assumption on $\mathcal{A}$ implies that α_n and β_n are invertible for large enough n. □

Fitting's lemma. Fix an abelian category.

Lemma 5.3 (Fitting)**.** *Let X be an object satisfying the bi-chain condition and ϕ an endomorphism.*

(1) *For large enough r, one has $X = \operatorname{Im}\phi^r \oplus \operatorname{Ker}\phi^r$.*
(2) *If X is indecomposable, then ϕ is either invertible or nilpotent.*

Proof. The endomorphism ϕ yields a bi-chain $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ with $X_n = \operatorname{Im}\phi^n$, $\alpha_n = \phi$, and β_n the inclusion. Because X satisfies the bi-chain condition, we may choose r large enough so that $\operatorname{Im}\phi^r = \operatorname{Im}\phi^{r+1}$. Thus $\phi^r \colon \operatorname{Im}\phi^r \to \operatorname{Im}\phi^{2r}$ is an isomorphism and we denote by ψ its inverse. Furthermore, let $\iota_1 \colon \operatorname{Im}\phi^r \to X$ and $\iota_2 \colon \operatorname{Ker}\phi^r \to X$ denote the inclusions. We put $\pi_1 = \psi\phi^r \colon X \to \operatorname{Im}\phi^r$ and $\pi_2 = \mathrm{id}_X - \psi\phi^r \colon X \to \operatorname{Ker}\phi^r$. Then $\iota_1\pi_1 + \iota_2\pi_2 = \mathrm{id}_X$ and $\pi_i\iota_i = \mathrm{id}_{X_i}$ for $i = 1, 2$. Thus $X = \operatorname{Im}\phi^r \oplus \operatorname{Ker}\phi^r$. Part (2) is an immediate consequence of (1). □

Proposition 5.4. *An object satisfying the bi-chain condition is indecomposable if and only if its endomorphism ring is local.*

Proof. Let X be an indecomposable object and ϕ, ϕ' a pair of endomorphisms. Suppose $\phi + \phi'$ is invertible, say $\rho(\phi + \phi') = \mathrm{id}_X$. If ϕ is non-invertible then $\rho\phi$ is non-invertible. Thus $\rho\phi$ is nilpotent, say $(\rho\phi)^r = 0$, by Lemma 5.3. We obtain

$$(\mathrm{id}_X - \rho\phi)(\mathrm{id}_X + \rho\phi + \ldots + (\rho\phi)^{r-1}) = \mathrm{id}_X .$$

Therefore $\rho\phi' = \mathrm{id}_X - \rho\phi$ is invertible whence ϕ' is invertible.

If $X = X_1 \oplus X_2$ with $X_i \neq 0$ for $i = 1, 2$, then we have idempotent endomorphisms ε_i of X with $\operatorname{Im}\varepsilon_i = X_i$. Clearly, each ε_i is non-invertible but $\mathrm{id}_X = \varepsilon_1 + \varepsilon_2$. □

Krull-Remak-Schmidt decompositions. Fix an abelian category.

Theorem 5.5 (Atiyah)**.** *An object satisfying the bi-chain condition admits a decomposition into a finite direct sum of indecomposable objects having local endomorphism rings.*

Proof. Fix an object X satisfying the bi-chain condition. Assume that X has no decomposition into a finite direct sum of indecomposable objects. Then there is a decomposition $X = X_1 \oplus Y_1$ such that X_1 has no decomposition into a finite direct sum of indecomposable objects and $Y_1 \neq 0$. We continue decomposing X_1 and obtain a bi-chain $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\beta_n} X_n$ $(n \geq 0)$ with $X = X_0$ and $\alpha_n\beta_n = \mathrm{id}_{X_{n+1}}$ for all $n \geq 0$. This bi-chain does not terminate and this is a contradiction.

It remains to observe that any direct summand of X satisfies the bi-chain condition. In particular, every indecomposable direct summand has a local endomorphism ring by Proposition 5.4. □

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