NOTES ON LOCAL COHOMOLOGY AND SUPPORT

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Abstract. These are notes for a course in Bielefeld in summer 2013. Most of the material is taken from joint work with Dave Benson and Srikanth Iyengar, in particular from [2].

1. Three examples from representation theory

We present three classical examples from representation theory. In each case there is a notion of support which provides a classification of all representations.

1.1. **Endomorphisms of vector spaces.** Let k be a field. We consider *endomorphisms* (V,ϕ). These are pairs consisting of a finite dimensional k-vector space V and an endomorphism $\phi\colon V \to V$. Alternatively, one may think of (V,ϕ) as a k-linear representation of the quiver consisting of a single vertex and one loop.

$$\begin{tikzcd} \circ \arrow[loop right] \end{tikzcd}$$

A non-zero endomorphism (V,ϕ) is *indecomposable* if V admits no proper decomposition into ϕ-invariant subspaces. There is an essentially unique decomposition $V=\bigoplus_{i=1}^r V_i$ into a direct sum of ϕ-invariant subspaces such that each $(V_i,\phi|_{V_i})$ is indecomposable.

A basic invariant is the *minimal polynomial* $p_\phi \in k[t]$; it is the unique monic polynomial such that the ideal generated by p_ϕ equals the kernel of the homomorphism

$$k[t] \longrightarrow \operatorname{End}_k(V), \quad t \mapsto \phi.$$

The polynomial ring $k[t]$ is a principal ideal domain. Thus monic irreducible polynomials correspond bijectively to non-zero prime ideals by taking a polynomial p to the ideal generated by p. Let $\operatorname{Spec} k[t]$ denote the set of prime ideals of $k[t]$.

The *support* of (V,ϕ) is by definition

$$\operatorname{Supp}(V,\phi)=\{\mathfrak{p}\in \operatorname{Spec} k[t] \mid p_\phi \in \mathfrak{p}\}.$$

Lemma 1.1.1. *If (V,ϕ) is indecomposable, then $\operatorname{Supp}(V,\phi)=\{\mathfrak{p}\}$ for some $\mathfrak{p}\in \operatorname{Spec} k[t]$.* □

Suppose that $\operatorname{Supp}(V,\phi)=\{\mathfrak{p}\}$. The *length* of (V,ϕ) is the number $n\ge 1$ such that $(p_\phi)=\mathfrak{p}^n$.

Proposition 1.1.2. *Two indecomposable endomorphisms are isomorphic if and only if they have the same support and the same length.* □

This is a preliminary version from April 15, 2013. Updates will be frequent.

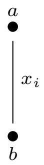
1.2. **Representations of the Klein four group.** Let

$$G = \langle g_1, g_2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

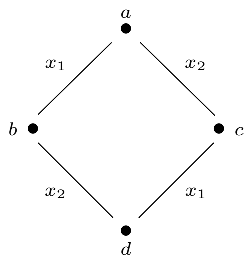
and let k be a field of characteristic two. Let kG be the group algebra of G over k, and let $x_1 = g_1 - 1$, $x_2 = g_2 - 1$ as elements of kG. Then $x_1^2 = x_2^2 = 0$, and we have

$$kG = k[x_1, x_2]/(x_1^2, x_2^2).$$

We describe kG-modules by diagrams in which the vertices represent basis elements as a k-vector space, and an edge

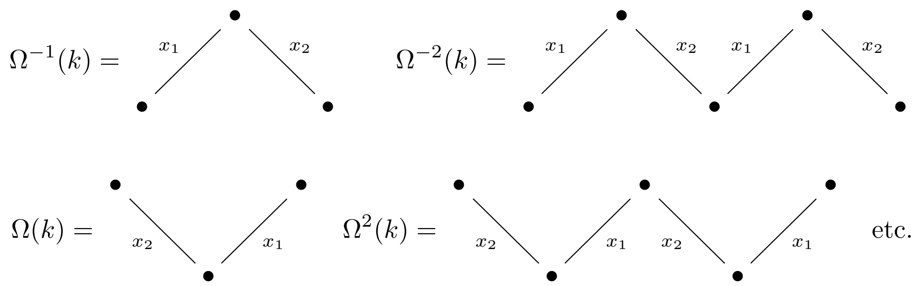


indicates that $x_i a = b$. If there is no edge labelled x_i in the downwards direction from a vertex then x_i sends the corresponding basis vector to zero. For example, the group algebra kG has the following diagram:



As a vector space, $kG = ka \oplus kb \oplus kc \oplus kd$. We have $\operatorname{rad}^2 kG = \operatorname{soc} kG = kd$, $\operatorname{rad} kG = \operatorname{soc}^2 kG = kb \oplus kc \oplus kd$.

Here are the diagrams for the syzygies of the trivial module:



For each integer $n \geq 0$ we have

$$\operatorname{Ext}^n_{kG}(k,k) \xrightarrow{\sim} \underline{\operatorname{Hom}}_{kG}(k, \Omega^{-n}(k)) \tag{1.2.1}$$

and so $\dim_k \operatorname{Ext}^n_{kG}(k,k) = n+1$. In fact, the full cohomology algebra is the $\mathbb{Z}$-graded algebra

$$H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k) = k[\zeta_1, \zeta_2]$$

with $\deg(\zeta_1) = \deg(\zeta_2) = 1$.

The ring $H^*(G,k)$ is a two-dimensional graded factorial domain. Thus homogeneous irreducible elements correspond to non-zero homogeneous prime ideals by taking an element p to the ideal generated by p. We write $\mathfrak{m} = H^+(G,k)$ for the unique maximal ideal consisting of positive degree elements. Let $\operatorname{Spec} H^*(G,k)$ denote the set of homogeneous prime ideals of $H^*(G,k)$.

1.3. **A classification of the representations of the Klein four group.** The finite dimensional indecomposable kG-modules come in three types [1, §4.3]:

(1) The group algebra kG itself.
(2) For each $n \in \mathbb{Z}$, the module $\Omega^n(k)$.
(3) For each $\mathfrak{p} \in \operatorname{Spec} H^*(G,k) \setminus \{0, \mathfrak{m}\}$ and $r \in \mathbb{N}$, a module $L_{\mathfrak{p}^r}$.

Let $\mathfrak{p} \in \operatorname{Spec} H^*(G,k) \setminus \{0, \mathfrak{m}\}$ and choose a homogeneous irreducible element p of degree d that generates $\mathfrak{p}$. The bijection (1.2.1) gives for each power p^r a monomorphism $k \to \Omega^{-rd}(k)$ whose cokernel we denote by $L_{\mathfrak{p}^r}$. Thus there is an exact sequence

$$0 \longrightarrow k \longrightarrow \Omega^{-rd}(k) \longrightarrow L_{\mathfrak{p}^r} \longrightarrow 0.$$

Given a finite dimensional kG-module M, consider the homomorphism

$$\chi_M \colon H^*(G,k) \longrightarrow \operatorname{Ext}^*_{kG}(M,M), \quad \eta \mapsto M \otimes_k \eta.$$

The *support* of M is by definition the set

$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} H^*(G,k) \mid \operatorname{Ker} \chi_M \subseteq \mathfrak{p}\}.$$

Proposition 1.3.1. *Let $\mathfrak{p} \in \operatorname{Spec} H^*(G,k) \setminus \{0, \mathfrak{m}\}$ and $n \in \mathbb{Z}$. Then we have*

$$\operatorname{Supp} kG = \{\mathfrak{m}\}, \quad \operatorname{Supp} \Omega^n(k) = \operatorname{Spec} H^*(G,k), \quad \operatorname{Supp} L_{\mathfrak{p}^n} = \{\mathfrak{p}, \mathfrak{m}\}. \qquad \square$$

1.4. **Coherent sheaves on $\mathbb{P}^1_k$.** Let k be a field and $\mathbb{P}^1_k$ the projective line over k. We view $\mathbb{P}^1_k$ as a scheme and begin with a description of the underlying set of points.

Let $k[x_0, x_1]$ be the polynomial ring in two variables with the usual $\mathbb{Z}$-grading by total degree. Denote by $\operatorname{Proj} k[x_0,x_1]$ the set of homogeneous prime ideals of $k[x_0,x_1]$ that are different from the unique maximal ideal consisting of positive degree elements. Note that $k[x_0,x_1]$ is a two-dimensional graded factorial domain. Thus homogeneous irreducible polynomials correspond to non-zero homogeneous prime ideals by taking a polynomial p to the ideal generated by p.

The elements of $\operatorname{Proj} k[x_0,x_1]$ form the *points* of $\mathbb{P}^1_k$. A point $\mathfrak{p} \in \mathbb{P}^1_k$ is *closed* if $\mathfrak{p} \neq 0$. Using homogeneous coordinates, a *rational point* of $\mathbb{P}^1_k$ is a pair $[\lambda_0 : \lambda_1]$ of elements of k which are not both zero, subject to the relation $[\lambda_0 : \lambda_1] = [\alpha\lambda_0 : \alpha\lambda_1]$ for all $\alpha \in k$, $\alpha \neq 0$. We identify each rational point $[\lambda_0 : \lambda_1]$ with the prime ideal $(\lambda_1 x_0 - \lambda_0 x_1)$ of $k[x_0,x_1]$. If k is algebraically closed then all closed points are rational.

Using the identification $y = x_1/x_0$, we cover $\mathbb{P}^1_k$ by two copies $U' = \operatorname{Spec} k[y]$ and $U'' = \operatorname{Spec} k[y^{-1}]$ of the affine line, with $U' \cap U'' = \operatorname{Spec} k[y, y^{-1}]$. More precisely, the morphism $k[x_0,x_1] \to k[y]$ which sends a polynomial p to $p(1,y)$ induces a bijection

$$\operatorname{Proj} k[x_0,x_1] \setminus \{(x_0)\} \xrightarrow{\sim} \operatorname{Spec} k[y].$$

Analogously, the morphism $k[x_0,x_1] \to k[y^{-1}]$ which sends a polynomial p to $p(y^{-1},1)$ induces a bijection

$$\operatorname{Proj} k[x_0,x_1] \setminus \{(x_1)\} \xrightarrow{\sim} \operatorname{Spec} k[y^{-1}].$$

Based on the covering $\mathbb{P}^1_k = U' \cup U''$, the category $\operatorname{\mathsf{coh}} \mathbb{P}^1_k$ of coherent sheaves admits a description in terms of the following pullback of abelian categories

$$\begin{tikzcd} \operatorname{\mathsf{coh}} \mathbb{P}^1_k \arrow[r] \arrow[d] & \operatorname{\mathsf{coh}} U' \arrow[d] \\ \operatorname{\mathsf{coh}} U'' \arrow[r] & \operatorname{\mathsf{coh}} U' \cap U'' \end{tikzcd}$$

where each functor is given by restricting a sheaf to the appropriate open subset; see [4, Chap. VI, Prop. 2]. More concretely, this pullback diagram has, up to equivalence, the form

$$\begin{tikzcd} \mathsf{A} \arrow[r] \arrow[d] & \operatorname{mod} k[y] \arrow[d] \\ \operatorname{mod} k[y^{-1}] \arrow[r] & \operatorname{mod} k[y, y^{-1}] \end{tikzcd}$$

where the category A is defined as follows. The objects of A are triples (M', M'', μ), where M' is a finitely generated $k[y]$-module, M'' is a finitely generated $k[y^{-1}]$-module, and $\mu\colon M'_y \xrightarrow{\sim} M''_{y^{-1}}$ is an isomorphism of $k[y, y^{-1}]$-modules. Here, we use for any R-module M the notation M_x to denote the localisation with respect to an element $x \in R$. A morphism from (M', M'', μ) to (N', N'', ν) in A is a pair (ϕ', ϕ'') of morphisms, where $\phi'\colon M' \to N'$ is $k[y]$-linear and $\phi''\colon M'' \to N''$ is $k[y^{-1}]$-linear such that $\nu\phi'_y = \phi''_{y^{-1}}\mu$.

Given a sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$, we denote for any open subset $U \subseteq \mathbb{P}^1_k$ by $\Gamma(U, \mathcal{F})$ the *sections* over U.

Lemma 1.4.1. *The assignment*

$$\mathcal{F} \longmapsto (\Gamma(U', \mathcal{F}), \Gamma(U'', \mathcal{F}), \mathrm{id}_{\Gamma(U' \cap U'', \mathcal{F})})$$

gives an equivalence $\operatorname{coh} \mathbb{P}^1_k \xrightarrow{\sim} \mathsf{A}$.

Proof. The description of a sheaf $\mathcal{F}$ on $\mathbb{P}^1_k = U' \cup U''$ in terms of its restrictions $\mathcal{F}|_{U'}$, $\mathcal{F}|_{U''}$, and $\mathcal{F}|_{U' \cap U''}$ is standard; see [4, Chap. VI, Prop. 2]. Thus it remains to observe that taking global sections identifies $\operatorname{coh} U' = \operatorname{mod} k[y]$, $\operatorname{coh} U'' = \operatorname{mod} k[y^{-1}]$, and $\operatorname{coh} U' \cap U'' = \operatorname{mod} k[y, y^{-1}]$. □

From now on we identify the categories $\operatorname{coh} \mathbb{P}^1_k$ and A via the above equivalence.

Let $\operatorname{grmod} k[x_0, x_1]$ denote the category of finitely generated $\mathbb{Z}$-graded $k[x_0, x_1]$-modules and let $\operatorname{grmod}_0 k[x_0, x_1]$ be the Serre subcategory consisting of all finite length modules.

There is a functor

$$\operatorname{grmod} k[x_0, x_1] \longrightarrow \operatorname{coh} \mathbb{P}^1_k \tag{1.4.2}$$

that takes a graded $k[x_0, x_1]$-module M to the triple

$$\widetilde{M} = ((M_{x_0})_0, (M_{x_1})_0, \sigma_M),$$

where the variable y acts on the degree zero part of M_{x_0} via the identification $y = x_1/x_0$, the variable y^{-1} acts on the degree zero part of M_{x_1} via the identification $y^{-1} = x_0/x_1$, and the isomorphism σ_M equals the obvious identification $[(M_{x_0})_0]_{x_1/x_0} = [(M_{x_1})_0]_{x_0/x_1}$. Note that this functor annihilates precisely the finite length modules.

Given an abelian category C and a Serre subcategory $\mathsf{D} \subseteq \mathsf{C}$, the *quotient category* C/D is obtained by formally inverting all morphisms in C such that kernel and cokernel belong to D [4, Chap. III].

Proposition 1.4.3 (Serre [5])**.** *The functor* (1.4.2) *induces an equivalence*

$$\frac{\operatorname{grmod} k[x_0, x_1]}{\operatorname{grmod}_0 k[x_0, x_1]} \xrightarrow{\sim} \operatorname{coh} \mathbb{P}^1_k. \qquad \square$$

1.5. **A classification of the coherent sheaves on $\mathbb{P}^1_k$.** For any $n \in \mathbb{Z}$ and $\mathcal{F} = (M', M'', \mu)$ in $\operatorname{coh} \mathbb{P}^1_k$, denote by $\mathcal{F}(n)$ the *twisted sheaf* $(M', M'', \mu^{(n)})$, where $\mu^{(n)}$ is the map μ followed by multiplication with y^{-n}. Given a graded $k[x_0, x_1]$-module M, the *twisted module* $M(n)$ is obtained by shifting the grading, that is, $M(n)_i = M_{i+n}$ for $i \in \mathbb{Z}$. Note that $\widetilde{M}(n) = \widetilde{M(n)}$.

The *structure sheaf* is the sheaf $\mathcal{O} = (k[y], k[y^{-1}], \mathrm{id}_{k[y,y^{-1}]})$; it is the image of the free $k[x_0, x_1]$-module of rank one under the functor (1.4.2). For any pair $m, n \in \mathbb{Z}$, we have a natural bijection

$$k[x_0, x_1]_{n-m} \xrightarrow{\sim} \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n)). \tag{1.5.1}$$

The map sends a homogeneous polynomial p of degree $n - m$ to the morphism (ϕ', ϕ''), where $\phi' \colon k[y] \to k[y]$ is multiplication by $p(1, y)$ and $\phi'' \colon k[y^{-1}] \to k[y^{-1}]$ is multiplication by $p(y^{-1}, 1)$.

Each coherent sheaf $\mathcal{F}$ admits an essentially unique decomposition $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{F}_i$ into indecomposable sheaves. The indecomposable sheaves come in two types:

(1) For each $n \in \mathbb{Z}$, the sheaf $\mathcal{O}(n)$.
(2) For each closed point $\mathfrak{p} \in \mathbb{P}^1_k$ and $r \in \mathbb{N}$, a sheaf $\mathcal{O}_{\mathfrak{p}^r}$.

Let $\mathfrak{p}$ be a closed point and choose a homogeneous irreducible polynomial p of degree d that generates $\mathfrak{p}$. The bijection (1.5.1) gives for each power p^r a monomorphism $\mathcal{O} \to \mathcal{O}(rd)$ whose cokernel we denote by $\mathcal{O}_{\mathfrak{p}^r}$. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(rd) \longrightarrow \mathcal{O}_{\mathfrak{p}^r} \longrightarrow 0.$$

Given a sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$ and a point $\mathfrak{p} \in \mathbb{P}^1_k$, the *stalk* of $\mathcal{F}$ at $\mathfrak{p}$ is the colimit

$$\mathcal{F}_{\mathfrak{p}} = \operatorname{colim}_{\mathfrak{p} \in U} \mathcal{F}(U)$$

where U runs through all open subsets of $\mathbb{P}^1_k$. The *support* of $\mathcal{F}$ is by definition

$$\operatorname{Supp} \mathcal{F} = \{\mathfrak{p} \in \mathbb{P}^1_k \mid \mathcal{F}_{\mathfrak{p}} \neq 0\}.$$

The functor (1.4.2) provides an alternative description of the support. In fact, for each graded $k[x_0, x_1]$-module M and $\mathfrak{p} \in \mathbb{P}^1_k$, the functor induces an isomorphism

$$(M_{\mathfrak{p}})_0 \xrightarrow{\sim} (\widetilde{M})_{\mathfrak{p}}.$$

Composing the natural homomorphism

$$k[x_0, x_1] \longrightarrow \operatorname{End}^*(M) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}(M, M(n))$$

with the induced homomorphism $\operatorname{End}^*(M) \to \operatorname{End}^*(\widetilde{M})$ yields for each $\mathcal{F}$ in $\operatorname{coh} \mathbb{P}^1_k$ a homomorphism

$$\chi_{\mathcal{F}} \colon k[x_0, x_1] \longrightarrow \operatorname{End}^*(\mathcal{F}).$$

Lemma 1.5.2. *We have*

$$\operatorname{Supp} \mathcal{F} = \{\mathfrak{p} \in \mathbb{P}^1_k \mid \operatorname{Ker} \chi_{\mathcal{F}} \subseteq \mathfrak{p}\}. \qquad \square$$

Proposition 1.5.3. *Let $\mathfrak{p} \in \mathbb{P}^1_k$ be a closed point and $n \in \mathbb{Z}$. Then we have*

$$\operatorname{Supp} \mathcal{O}(n) = \mathbb{P}^1_k \qquad \textit{and} \qquad \operatorname{Supp} \mathcal{O}_{\mathfrak{p}^n} = \{\mathfrak{p}\}. \qquad \square$$

Remark 1.5.4. The sheaf $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1)$ is a tilting object and its endomorphism algebra is isomorphic to the Kronecker algebra Λ (i.e. the path algebra of the quiver $\circ \rightrightarrows \circ$). This yields a derived equivalence

$$\mathbf{R}\operatorname{Hom}(\mathcal{T}, -) \colon \mathsf{D}^b(\operatorname{coh} \mathbb{P}^1_k) \xrightarrow{\sim} \mathsf{D}^b(\operatorname{mod} \Lambda)$$

and therefore a notion of support for each Λ-module.

2. Support for modules over commutative rings

Let A be a commutative noetherian ring. We consider the category $\mathsf{Mod}\,A$ of A-modules and its full subcategory $\mathsf{mod}\,A$ which is formed by all finitely generated A-modules. Note that an A-module is finitely generated if and only if it is noetherian.

The *spectrum* $\operatorname{Spec} A$ of A is the set of prime ideals in it. A subset of $\operatorname{Spec} A$ is *Zariski closed* if it is of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some ideal $\mathfrak{a}$ of A. A subset $\mathcal{V}$ of $\operatorname{Spec} A$ is *specialisation closed* if for any pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals, $\mathfrak{p} \in \mathcal{V}$ implies $\mathfrak{q} \in \mathcal{V}$.

2.1. **Support.** The *support* of an A-module M is the subset

$$\operatorname{Supp}_A M = \{\mathfrak{p} \in \operatorname{Spec} A \mid M_\mathfrak{p} \neq 0\}\,.$$

Observe that this is a specialisation closed subset of $\operatorname{Spec} A$.

Lemma 2.1.1. *One has* $\operatorname{Supp}_A A/\mathfrak{a} = \mathcal{V}(\mathfrak{a})$ *for each ideal* $\mathfrak{a}$ *of* A.

Proof. Fix $\mathfrak{p} \in \operatorname{Spec} A$ and let $S = A \setminus \mathfrak{p}$. Recall that for any A-module M, an element x/s in $S^{-1}M = M_\mathfrak{p}$ is zero iff there exists $t \in S$ such that $tx = 0$. Thus we have $(A/\mathfrak{a})_\mathfrak{p} = 0$ iff there exists $t \in S$ with $t(1+\mathfrak{a}) = t + \mathfrak{a} = 0$ iff $\mathfrak{a} \not\subseteq \mathfrak{p}$. □

Lemma 2.1.2. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of* A*-modules, then* $\operatorname{Supp}_A M = \operatorname{Supp}_A M' \cup \operatorname{Supp}_A M''$.

Proof. The sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is exact for each $\mathfrak{p}$ in $\operatorname{Spec} A$. □

Lemma 2.1.3. *Let* $M = \sum_i M_i$ *be an* A*-module, written as a sum of submodules* M_i. *Then* $\operatorname{Supp}_A M = \bigcup_i \operatorname{Supp}_A M_i$.

Proof. The assertion is clear if the sum $\sum_i M_i$ is direct, since

$$\bigoplus_i (M_i)_\mathfrak{p} = \big(\bigoplus_i M_i\big)_\mathfrak{p}.$$

As $M_i \subseteq M$ for all i one gets $\bigcup_i \operatorname{Supp}_A M_i \subseteq \operatorname{Supp}_A M$, from Lemma 2.1.2. On the other hand, $M = \sum_i M_i$ is a factor of $\bigoplus_i M_i$, so $\operatorname{Supp}_A M \subseteq \bigcup_i \operatorname{Supp}_A M_i$. □

We write $\operatorname{Ann}_A M$ for the ideal of elements in A that annihilate M; it is the kernel of the natural homomorphism

$$A \longrightarrow \operatorname{End}_A(M).$$

Lemma 2.1.4. *One has* $\operatorname{Supp}_A M \subseteq \mathcal{V}(\operatorname{Ann}_A M)$, *with equality when* M *is in* $\mathsf{mod}\,A$.

Proof. Write $M = \sum_i M_i$ as a sum of cyclic modules $M_i \cong A/\mathfrak{a}_i$. Then

$$\operatorname{Supp}_A M = \bigcup_i \operatorname{Supp}_A M_i = \bigcup_i \mathcal{V}(\mathfrak{a}_i) \subseteq \mathcal{V}\big(\bigcap_i \mathfrak{a}_i\big) = \mathcal{V}(\operatorname{Ann}_A M),$$

and equality holds if the sum is finite. □

Lemma 2.1.5. *Let* $M \neq 0$ *be an* A*-module. If* $\mathfrak{p}$ *is maximal in the set of ideals which annihilate a non-zero element of* M*, then* $\mathfrak{p}$ *is prime.*

Proof. Suppose $0 \neq x \in M$ and $\mathfrak{p}x = 0$. Let $a, b \in A$ with $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$. Then $(\mathfrak{p}, b)$ annihilates $ax \neq 0$, so the maximality of $\mathfrak{p}$ implies $b \in \mathfrak{p}$. Thus $\mathfrak{p}$ is prime. □

Lemma 2.1.6. *Let* $M \neq 0$ *be an* A*-module. There exists a submodule of* M *which is isomorphic to* $A/\mathfrak{p}$ *for some prime ideal* $\mathfrak{p}$.

Proof. The set of ideals annihilating a non-zero element has a maximal element, since A is noetherian. Now apply Lemma 2.1.5. □

Lemma 2.1.7. *For each M in $\operatorname{\mathsf{mod}} A$ there exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

such that each factor M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. In that case one has $\operatorname{Supp}_A M = \bigcup_i \mathcal{V}(\mathfrak{p}_i)$.

Proof. Repeated application of Lemma 2.1.6 yields a chain of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of M such that each M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some $\mathfrak{p}_i$. This chain stabilises since M is noetherian, and therefore $\bigcup_i M_i = M$.

The last assertion follows from Lemmas 2.1.2 and 2.1.1. □

2.2. **Serre subcategories.** A full subcategory C of A-modules is called *Serre subcategory* if for every exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules, M belongs to C if and only if M' and M'' belong to C. We set

$$\operatorname{Supp}_A \mathsf{C} = \bigcup_{M \in \mathsf{C}} \operatorname{Supp}_A M.$$

Proposition 2.2.1. *The assignment $\mathsf{C} \mapsto \operatorname{Supp}_A \mathsf{C}$ induces a bijection between*

- *the set of Serre subcategories of $\operatorname{\mathsf{mod}} A$, and*
- *the set of specialisation closed subsets of $\operatorname{Spec} A$.*

Its inverse takes $\mathcal{V} \subseteq \operatorname{Spec} A$ to $\{M \in \operatorname{\mathsf{mod}} A \mid \operatorname{Supp} M \subseteq \mathcal{V}\}$.

Proof. Both maps are well defined by Lemmas 2.1.2 and 2.1.4. If $\mathcal{V} \subseteq \operatorname{Spec} A$ is a specialisation closed subset, let $\mathsf{C}_{\mathcal{V}}$ denote the smallest Serre subcategory containing $\{A/\mathfrak{p} \mid \mathfrak{p} \in \mathcal{V}\}$. Then we have $\operatorname{Supp} \mathsf{C}_{\mathcal{V}} = \mathcal{V}$, by Lemmas 2.1.1 and 2.1.2. Now let C be a Serre subcategory of $\operatorname{\mathsf{mod}} A$. Then

$$\operatorname{Supp} \mathsf{C} = \{\mathfrak{p} \in \operatorname{Spec} A \mid A/\mathfrak{p} \in \mathsf{C}\}$$

by Lemma 2.1.7. It follows that $\mathsf{C} = \mathsf{C}_{\mathcal{V}}$ for each Serre subcategory C, where $\mathcal{V} = \operatorname{Supp} \mathsf{C}$. Thus $\operatorname{Supp} \mathsf{C}_1 = \operatorname{Supp} \mathsf{C}_2$ implies $\mathsf{C}_1 = \mathsf{C}_2$ for each pair $\mathsf{C}_1, \mathsf{C}_2$ of Serre subcategories. □

Corollary 2.2.2. *Let M and N be in $\operatorname{\mathsf{mod}} A$. Then $\operatorname{Supp}_A N \subseteq \operatorname{Supp}_A M$ if and only if N belongs to the smallest Serre subcategory containing M.*

Proof. With C denoting the smallest Serre subcategory containing M, there is an equality $\operatorname{Supp}_A \mathsf{C} = \operatorname{Supp}_A M$ by Lemma 2.1.2. Now apply Proposition 2.2.1. □

2.3. **Localising subcategories.** A full subcategory C of A-modules is said to be *localising* if it is a Serre subcategory and if for any family of A-modules $M_i \in \mathsf{C}$ the sum $\bigoplus_i M_i$ is in C.

Corollary 2.3.1 (Gabriel [4])**.** *The assignment $\mathsf{C} \mapsto \operatorname{Supp}_A \mathsf{C}$ gives a bijection between*

- *the set of localising subcategories of $\operatorname{\mathsf{Mod}} A$, and*
- *the set of specialisation closed subsets of $\operatorname{Spec} A$.*

Its inverse takes $\mathcal{V} \subseteq \operatorname{Spec} A$ to $\{M \in \operatorname{\mathsf{Mod}} A \mid \operatorname{Supp}_A M \subseteq \mathcal{V}\}$.

Proof. The proof is essentially the same as the one of Proposition 2.2.1 if we observe that any A-module M is the sum $M = \sum_i M_i$ of its finitely generated submodules. Note that M belongs to a localising subcategory C if and only if all M_i belong to C. In addition, we use that $\operatorname{Supp}_A M = \bigcup_i \operatorname{Supp}_A M_i$; see Lemma 2.1.3. □

2.4. **Graded rings and modules.** The results in this section generalise to graded modules over graded rings. We sketch the appropriate setting, following closely the exposition in [3].

Fix an abelian *grading group* G endowed with a symmetric bilinear form

$$(-,-)\colon G \times G \longrightarrow \mathbb{Z}/2.$$

We consider a ring A with a decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that the multiplication satisfies $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$. We say that A is *G-graded commutative* when $xy = (-1)^{(g,h)} yx$ for all homogeneous elements $x \in A_g$, $y \in A_h$. A homogeneous element in A is *even* if it belongs to A_g for some $g \in G$ satisfying $(g, h) = 0$ for all $h \in G$.

Let us fix such a G-graded commutative ring A. We consider graded A-modules and homogeneous ideals of A. Note that all homogeneous ideals are automatically two-sided. The graded localisation of A at a multiplicative set consisting of even (and therefore central) homogeneous elements is the obvious one and enjoys the usual properties; in particular, it is again a G-graded commutative ring. Similarly, one localises any graded A-module at such a multiplicative set. For instance, when $\mathfrak{p}$ is a homogeneous prime ideal of A and M is a graded A-module, then $M_\mathfrak{p}$ is the localisation of M with respect to the multiplicative set of even homogeneous elements in $A \setminus \mathfrak{p}$.

Suppose now that A is *noetherian* as a G-graded ring, that is, the ascending chain condition holds for homogeneous ideals of A. Then all results of this section carry over to the category of graded A-modules. However, it is necessary to twist. Recall that for any graded A-module M and $g \in G$, the *twist* $M(g)$ is the A-module M with the new grading defined by $M(g)_h = M_{g+h}$ for each $h \in G$. For instance, in Lemma 2.1.6 one shows that each graded non-zero module has a submodule of the form $(A/\mathfrak{p})(g)$ for some homogeneous prime ideal $\mathfrak{p}$ and some $g \in G$. This affects all subsequent statements. For example, Proposition 2.2.1 then classifies the Serre subcategories that are closed under twists.

References

[1] D. J. Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics, 30, Cambridge Univ. Press, Cambridge, 1991.

[2] D. J. Benson, S. Iyengar and H. Krause, *Representations of finite groups: local cohomology and support*, Oberwolfach Seminars, 43, Birkhäuser/Springer Basel AG, Basel, 2012.

[3] I. Dell'Ambrogio and G. Stevenson, On the derived category of a graded commutative Noetherian ring, J. Algebra **373** (2013), 356–376.

[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France **90** (1962), 323–448.

[5] J.-P. Serre, Faisceaux algébriques cohérents, Ann. of Math. (2) **61** (1955), 197–278.

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