$\boxed{A\,61a}$ $f: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = x^3 + y^3 + 3xy$

$\boxed{1}$ $0 \overset{!}{=} \operatorname{grad} f(x,y) = (3x^2+3y\,,\ 3y^2+3x) \iff x^2+y=0 \wedge y^2+x=0$

$\iff x^2+y=0 \wedge x=-y^2 \iff (-y^2)^2+y=0 \wedge x=-y^2$

$\iff y^4+y=0 \wedge x=-y^2 \iff (y=0 \vee y^3=-1) \wedge x=-y^2$

$\iff \big((y=0 \wedge x=0) \vee (y=-1 \wedge x=-1) \big)$

Kritische Punkte : $P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$\boxed{2}$ $Hf(x,y) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$ $\boxed{2.1}$ $Hf(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ Eigenwerte von $Hf(0,0)$;

$0 = \begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - 9 \Rightarrow \lambda = \pm 3 \Rightarrow Hf(0,0)$ indefinit $\Rightarrow$ SP bei $(0,0)$

$f(0,0) = 0$

$\boxed{2.2}$ $Hf(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$, det $-6 < 0$, $\det Hf(-1,-1) = 36 - 9 > 0 \Rightarrow$

neg df $\Rightarrow$ Maximum $(-1,-1)$, $f(-1,-1) = 1$

$\boxed{61\,b}$ $\quad g(x,y,z) = x^2 + y^2 + z^2 - 2xyz$

$\boxed{1}$ $\quad 0 \overset{!}{=} \operatorname{grad} g(x,y,z) = (2x - 2yz,\ 2y - 2xz,\ 2z - 2xy) \Rightarrow$

$x - yz = 0 \wedge y - xz = 0 \wedge z - xy = 0 \Leftrightarrow \cancel{x = yz \wedge y = yz^2 \wedge z = xy}$

$\Rightarrow \cancel{x = yz \wedge y = yx^2y^2 \wedge z = xy} \Leftrightarrow x = yz \wedge y = xz \wedge z = xy$

$\Leftrightarrow x = yz \wedge y = yz^2 \wedge z = yzy \Leftrightarrow x = yz \wedge y = yz^2 \wedge z = zy^2$

$\Leftrightarrow x = yz \wedge y(1 - z^2) = 0 \wedge z(1 - y^2) = 0$

$\Leftrightarrow x = yz \wedge (y = 0 \vee z^2 = 1) \wedge (z = 0 \vee y^2 = 1)$

$\Leftrightarrow \cancel{x = yz \wedge} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \hat{=}$ kritische Punkte

$\boxed{2}$ $\quad Hg(x,y,z) = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}$ $\quad$ 12.1

$\boxed{2.1}$ $H_g(0,0,0) = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$ pd $\Rightarrow$ Minimum bei $(0,0,0)$ mit Wert 0

$\boxed{2.2}$ $H_g(1,1,1) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$, $|2| = 2$, $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 8$, $|H_g(-1,-1,-1)| = \begin{vmatrix} 2 & -2 & -2 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{vmatrix} = 0$

$\begin{vmatrix} 2-\lambda & -2 & -2 \\ -2 & 2-\lambda & -2 \\ -2 & -2 & 2-\lambda \end{vmatrix} \Rightarrow$ $\lambda_1 = 4 \Rightarrow 4\lambda_2\lambda_3 \overset{?}{=} \det H \overset{?}{=} -32$ $\Rightarrow$ $\lambda_2\lambda_3 = -8$

$6 = 4 + \lambda_2 + \lambda_3$ $\qquad \lambda_2 + \lambda_3 = 2$

$\Rightarrow \lambda_1 = 4,\ \lambda_2 = 4,\ \lambda_3 = 2 \Rightarrow$ indf $\Rightarrow$ SP

$\boxed{2.3}$ $H_g(-1,1,-1) = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{matrix} 2 & 2 & -2 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{matrix} \Rightarrow \det = -32$

$\lambda_1 = 4 \Rightarrow \lambda_2 = 4,\ \lambda_3 = -2$

$\boxed{2.4}$ $H_g(1,-1,-1) = \begin{pmatrix} 2 & 2 & 2 \\ +2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \rightsquigarrow \begin{matrix} 2 & 2 & 2 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{matrix} \Rightarrow \det = -32$, $\lambda_1 = 4, \lambda_2 = 4$

$\lambda_3 = -2$

$\boxed{2.5}$ $H_g(-1,-1,1)$ genauso $\rightarrow$ $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = -2$