82) $f(x,y) = 4x^2 - 3xy$ $\boxed{1}$ $0 \stackrel{!}{=} \nabla f \Leftrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8x - 3y \\ -3x \end{pmatrix}$

$\Leftrightarrow \quad x = y = 0 \quad \boxed{2} \quad (0,0) \in \mathring{K} : \quad Hf(x,y) = \begin{pmatrix} 8 & -3 \\ -3 & 0 \end{pmatrix} = Hf(0,0,0)$

$0 \stackrel{!}{=} \begin{vmatrix} 8-\lambda & -3 \\ -3 & -\lambda \end{vmatrix} = -\lambda(8-\lambda) + 9 \Rightarrow \lambda = +9 \vee \lambda = -1 \Rightarrow Hf(0,0,0)$

indefinit $\Rightarrow$ SP in $(0,0,0)$ $\quad \boxed{3}$ $\partial K$ liefert NB

$g(x,y) := x^2 + y^2 - 1 = 0$ ⊛ und Lagrange Ansatz $\nabla f(x,y) = \lambda \nabla g(x,y)$

$\Leftrightarrow \quad (8x - 3y, -3x) = \lambda(2x, 2y) \quad \Leftrightarrow \quad 8x - 3y = 2\lambda x, \; -3x = 2\lambda y,$

$x^2 + y^2 - 1 = 0 \quad \Leftrightarrow \quad (8 - 2\lambda)x - 3y = 0, \; x = -\frac{2}{3}\lambda y = 0, \; x^2 + y^2 - 1 = 0$

$\Leftrightarrow (8 - 2\lambda)\left(-\frac{2}{3}\lambda y\right) - 3y = 0, \; x = -\frac{2}{3}\lambda y \quad \frac{4}{9}\lambda^2 y^2 + y^2 - 1 = 0$

$\Leftrightarrow y\left(-\frac{16}{3}\lambda + \frac{4}{3}\lambda^2 - 3\right) = 0, \; x = -\frac{2}{3}\lambda y, \; y^2\left(\frac{4}{9}\lambda^2 + 1\right) - 1 = 0$

$\Leftrightarrow \left(y = 0 \vee \lambda^2 - 4\lambda - \frac{9}{4} = 0\right) \wedge x = -\frac{2}{3}\lambda y \wedge y^2\left(\frac{4}{9}\lambda^2 + 1\right) - 1 = 0$

$$\Leftrightarrow \Big[\big(y=0 \wedge x=0 \wedge \lightning\big)\Big] \vee \Big[\big(\lambda=-\tfrac{1}{2} \vee \lambda=\tfrac{9}{2}\big) \wedge x=-\tfrac{2}{3}\lambda y \wedge y^2\big(\tfrac{4}{9}\lambda^2+1\big)-1=0\Big]$$

$$\Leftrightarrow \lambda=-\tfrac{1}{2},\ x=\tfrac{1}{3}y,\ y^2\big(\overbrace{\tfrac{1}{9}+1}^{10/9}\big)=1 \quad \vee \quad \lambda=\tfrac{9}{2},\ x=-3y,\ y^2(10)=1$$

$$\Leftrightarrow \lambda=-\tfrac{1}{2}\ x=\pm\tfrac{1}{\sqrt{10}},\ y=\pm\tfrac{3}{\sqrt{10}} \quad \vee \quad \lambda=\tfrac{9}{2},\ x=\mp\tfrac{3}{\sqrt{10}},\ y=\pm\tfrac{1}{\sqrt{10}}$$

$$\Leftrightarrow (x,y)\in\Big\{\tfrac{1}{\sqrt{10}}(1,3),\ \tfrac{1}{\sqrt{10}}(-1,-3),\ \tfrac{1}{\sqrt{10}}(-3,1),\ \tfrac{1}{\sqrt{10}}(3,-1)\Big\} =: \{P_1,\dots,P_4\}$$

$$f(P_1)=f(P_2)=-\tfrac{1}{2},\quad f(P_3)=f(P_4)=\tfrac{9}{2}$$

Auf $\partial K$ ist also $-\frac{1}{2}$ Minimum bei $P_1, P_2$ und $\frac{9}{2}$ Max bei $P_3$ und $P_4$. Da $g$ auf $\mathring{K}$ keine lok. Ex gibt sind das alle!

---

$\circledast$ wg $\operatorname{rang}\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = \operatorname{rg}(2x, 2y) = 1$ f alle $(x,y)$ mit $g(x,y)=0$

83) a) $R \subset \mathbb{R}^3$ 2-dim UMF $\Leftrightarrow$ ① $\forall a \in R \; \exists U \subset \mathbb{R}^3$ :

$f \circ h|_U : U \to \mathbb{R}$ ist $C^1$ ② $R \cap U = \{x \in U \mid f \circ h(x) = 0\}$

③ $\operatorname{rang}\left( \frac{\partial}{\partial x}(f\circ h)(x,y,z), \frac{\partial}{\partial y}(f\circ h)(x,y,z), \frac{\partial}{\partial z}(f\circ h)(x,y,z) \right) = 3-2 = 1 \quad \forall (x,y,z) \in R \cap U$

Betrachte :
$$\mathbb{R}^3 \xrightarrow{h} (\mathbb{R}^2) \supset (0,\infty) \times \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \sqrt{x^2+y^2} \\ z \end{pmatrix}$$

<u>Zu ①</u>: $f \in C^1, h \in C^1 \Rightarrow f \circ h \in C^1$ ($\forall U \subset \mathbb{R}^n$ offen)

<u>zu ②</u>: „$\subset$" $\vec{x} \in R \cap U \Rightarrow x \in U \wedge 0 = f(\sqrt{x^2+y^2}, z) = f(h(x,y,z))$

„$\supset$" $(x,y,z) \in U \wedge f(\sqrt{x^2+y^2}, z) = 0 \Rightarrow (x,y,z) \in U \cap R$

<u>zu ③</u>: $\frac{\partial}{\partial z} D((f\circ h)(x,y,z)) = D_z\, Df(h(x,y,z)) \cdot Dh(x,y,z)$

$= \left( \partial_1 f(h(x,y,z)) \quad \partial_2 f(h(x,y,z)) \right) \cdot \begin{pmatrix} \frac{2x}{\sqrt{\;}} & \frac{y}{\sqrt{\;}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \quad \frac{\partial}{\partial z}(f \circ h)(x,y,z) = \partial_2 f(h(x,y,z))$

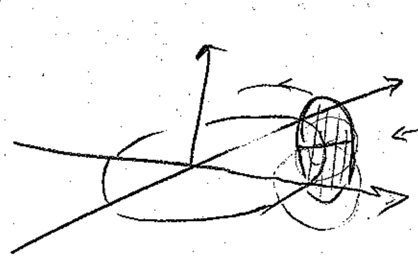
$= \Big( \partial_1 f(h(x,y,z)), \; \partial_1 f(h(x,y,z)), \; \partial_2 f(h(x,y,z)) \Big) \neq (0,0,0)$

denn da $M = f^{-1}(0)$ 1-dim Mf ist, ist

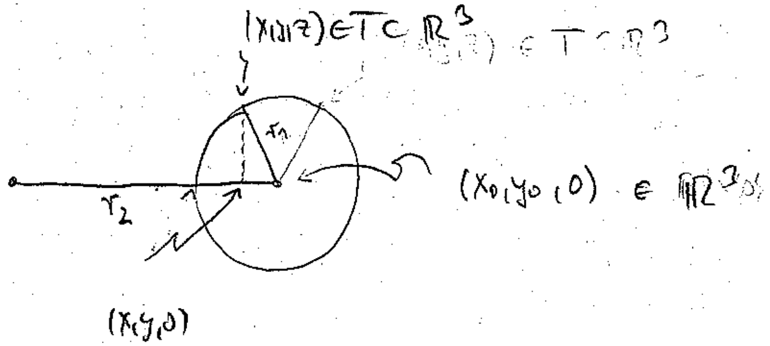
$\text{rang}\big(\partial_1 f(a,b), \partial_2 f(a,b)\big) = 2-1 = 1$ f. alle $(a,b)$ mit $f(a,b)$

$= 0$.

83b)




Rotation ~~des~~ vom $S^1$ um $y$-Achse $=$ Torus




$(x,y,z) \in T \Leftrightarrow$

$\boxed{1}$ $\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \right| = r_1$

$\boxed{2}$ $\left| \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right| = r_2$

$\boxed{3}$ $k \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $k > 0$

$\Leftrightarrow \boxed{2} \wedge \boxed{3} \wedge \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} kx \\ ky \\ 0 \end{pmatrix} \right| = r_1 \Leftrightarrow \boxed{3}, \left| \begin{pmatrix} kx \\ ky \\ 0 \end{pmatrix} \right| = r_2 \wedge \left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} kx \\ ky \\ 0 \end{pmatrix} \right| = r_1$

$\Leftrightarrow \boxed{3} \wedge k^2 = \frac{r_2^2}{x^2+y^2} \wedge (x-kx)^2 + (y-ky)^2 + z^2 = r_1^2$

$\Leftrightarrow \boxed{3} \wedge k = \frac{r_2}{\sqrt{x^2+y^2}} \wedge (x^2+y^2)\left(1 - \frac{r_2}{\sqrt{x^2+y^2}}\right) + z^2 = r_1^2$

Also $f(a,b) := a^2\left(1 - \frac{r_2}{a}\right) + b^2 - r_1^2$ in 83a)

8(a) $\frac{dx}{dt} = R \cdot (M-x)x \Rightarrow \frac{dx}{(M-x)x} = R\,dt$

$x'(t) \approx x(t)\,(M - x(t))$

Wachstum ↑ Bestand ↑ Restkapazität ↑ obere Schranke ↓

$\Rightarrow \int \frac{dx}{(M-x)x} = \int R\,dt = tR + C$

$\int_{t_0}^{x} \frac{dx}{}$

$\frac{A}{M-x} + \frac{B}{x} = \frac{1}{(M-x)x} \Leftrightarrow Ax + B(M-x) = 1 \Rightarrow \underline{x=0}: B = \frac{1}{M}$

$\underline{x=M}: A = \frac{1}{M} \Rightarrow \int \frac{dx}{(M-x)x} = \int \frac{1/M}{M-x}\,dx + \int \frac{1/M}{x}\,dx = -\frac{1}{M} \ln|M-x|$

$+ \frac{1}{M} \ln|x| = -\frac{1}{M} \ln\left(\left|\frac{M-x}{x}\right|\right)$

$\Rightarrow -\frac{1}{M} \ln\left(\left|\frac{M}{x}-1\right|\right) = tR + C \Rightarrow \ln\left(\left|\frac{M}{x}-1\right|\right) = -tMR - MC$

$\Rightarrow \left|\frac{M}{x}-1\right| = e^{-M(tR+C)} \underset{\text{da } M \text{ ob. Schranke}}{\overset{M \geq x(t)}{\Rightarrow}} \frac{M}{x} - 1 = (\pm) e^{-M(tR+C)}$

$\Rightarrow x = M \cdot \frac{1}{(\pm) e^{-M(tR+C)} + 1}$, $x(0) = M \cdot \frac{1}{(\pm) e^{-MC} + 1} \Rightarrow \frac{M}{x(0)} - 1 = (\pm) e^{-MC}$

$\Rightarrow M > x(0)$: $-MC = \ln\left(\frac{M}{x(0)} - 1\right) \Rightarrow C = -\frac{1}{M} \ln\left(\frac{M}{x(0)} - 1\right) \Rightarrow x(t) = \frac{M}{e^{-MtR}\left(\frac{M}{x(0)} - 1\right) + 1}$

Zu 84a $\quad g(t) := R \quad , \quad h(x(t)) = (m - x(t))\, x(t)$

TDV

$$\Rightarrow \quad G(t) = \int_{t_0}^{t} R \, d\xi = \int_{x_0}^{x(t)} (m - \cancel{x(t)}\,\mu)\, \cancel{x(t)}\,\mu \; d\mu$$

$\Rightarrow$ nach $x(t)$ auflösen

84b)

$$x(t) = M \cdot \frac{1}{1 + e^{-MtR}\left(\frac{M}{x(0)} - 1\right)}$$

$x(0) = 6,93 \quad (2010)$

$x(-50) = 3 \quad (1960)$

$M = 10$ obere Schranke

$$\Rightarrow \quad x(-50) = 10 \cdot \frac{1}{1 + e^{-10\,(-50)\cdot R}\left(\frac{10}{6,93} - 1\right)} \Rightarrow R \approx 0,0032$$

Wann 90%:

$$9 = x(t) = \frac{10}{1 + e^{-10\,t \cdot 0,0032}\left(\frac{10}{6,93} - 1\right)} \Rightarrow t \approx 2051,66$$

85a) $v: \mathbb{R}^3 \to \mathbb{R}^3 \quad x \mapsto w \times x = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} w_2 x_3 - w_3 x_2 \\ w_3 x_1 - w_1 x_3 \\ w_1 x_2 - w_2 x_1 \end{pmatrix}$

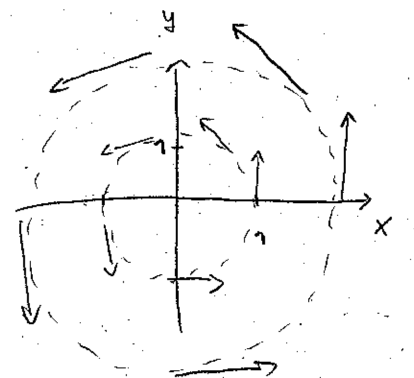
$$Dv(x_1, x_2, x_3) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

$$\operatorname{div} v(x_1, x_2, x_3) = \operatorname{spur} Dv(x_1, x_2, x_3) = 0$$

$$\operatorname{rot} v(x_1, x_2, x_3) = \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} w_1 + w_1 \\ -(-w_2 - w_2) \\ w_3 - (-w_3) \end{pmatrix} = 2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

<u>Bsp:</u> $\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $v(x) = w \times x$ <u>senkrecht</u> auf $w$ <u>und</u> $x$, $\|w \times x\|$ = Fläche Parallelogramm

<u>x-y-Ebene:</u>




genauso in jeder $(x\text{-}y\text{-}z)$-Ebene, da

$\|w \times \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\| = \|w \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}\|$

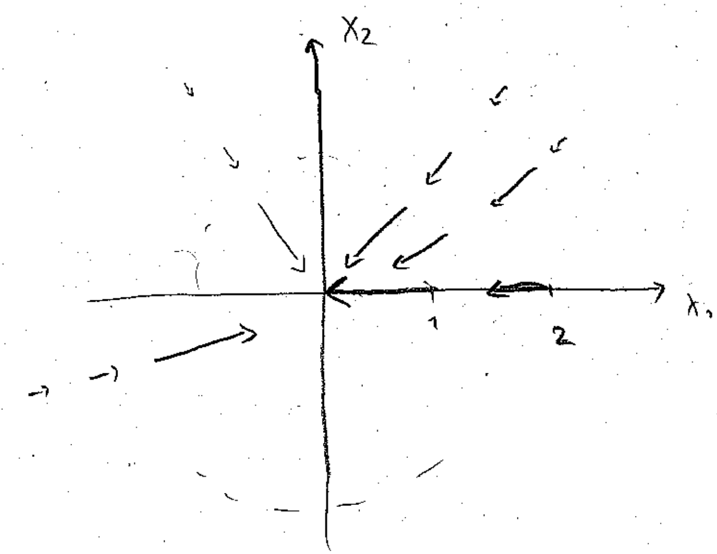
$\boxed{\text{oE } M = G = 1}$

85) $F(x_1, x_2, x_3) = - \frac{}{\|\vec{x}\|^3} \vec{x}$ , $\frac{\partial}{\partial x_i} \left( \|\vec{x}\|^{-3} \right) = -3 \, \|\vec{x}\|^{-4} \cdot \frac{2x_i}{2\|\vec{x}\|}$

$= - \frac{3x_i}{\|\vec{x}\|^5}$ , $\frac{\partial}{\partial x_j} F_j(x_1, x_2, x_3) \overset{PR}{=} + \frac{3x_i}{\|\vec{x}\|^5} \cdot x_j + \delta_{ij} \frac{-1}{\|\vec{x}\|^3}$

$\operatorname{div} F(x_1 x_2 x_3) = \sum_{i=1}^{3} \frac{3(x_i)^2}{\|\vec{x}\|^5} - \frac{\|x\|^2}{\|x\|^5} = -\left( \frac{3(x_1^2 + x_2^2 + x_3^2)}{\|\vec{x}\|^5} - \frac{3\|x\|^2}{\|x\|^5} \right) = 0$

$\underline{x_3 = 0}$: $F(\vec{x}) = - \frac{1}{\|x\|^3} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$




Bei (0,0,0) ist F nicht definiert!

$\operatorname{rot} F(x_1, x_2, x_3) = 0$

jeweils Differenz bilden!

$DF(x_1 x_2 x_3) = \frac{3}{\|\vec{x}\|^5} \begin{pmatrix} 0 & x_2 x_1 & x_3 x_1 \\ x_1 x_2 & 0 & x_3 x_2 \\ x_1 x_3 & x_2 x_3 & 0 \end{pmatrix}$