$\boxed{\text{Aufgabe 3}}$ Bch $\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$, $\quad u\in L^p(\mathbb{R}^d)$, $v\in L^q(\mathbb{R}^d)$, $\quad 1<p<r$, $1<q<r$

$\Rightarrow \quad \|u*v\|_r \le \|u\|_p \, \|v\|_q \qquad$ (Young'sche Ungleichung)

Beweis: $\boxed{1}$ Für $p_3=r$, $p_2=\frac{q}{1-q/r}$, $p_1=\frac{p}{1-p/r}$, $p=1$ gilt

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}+\frac{1-q/r}{q}+\frac{1-p/r}{p}=\frac{\cancel{qp}+rp-pq+rq-\cancel{pq}}{rqp}=\frac{1}{q}-\frac{1}{r}+\frac{1}{p}$$

$= 1 = \frac{1}{p}$ also die verallg. Höldersche Ungleichung

$\|f_1f_2f_3\|_1 \le \|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}$ für $f_i\in L^{p_i}(\mathbb{R})$ und $f_1f_2f_3\in L^1$

$\boxed{2}$ $$\underbrace{|u(x-y)\,v(y)|}_{f} = \underbrace{|u(x-y)|^{1-p/r}}_{f_1}\cdot\underbrace{|v(y)|^{1-q/r}}_{f_2}\cdot\underbrace{\left(|u(x-y)|^p\,|v(y)|^q\right)^{1/r}}_{f_3}$$

$\boxed{3}$ $f_1\in L^{p_1}(\mathbb{R})$: $\|f_1\|_{p_1}^{p_1}=\int\left(|u(x-y)|^{1-p/r}\right)^{p_1}dx=\int|u(x-y)|^p\,dx=\|u\|_p^p$

$f_2\in L^{p_2}(\mathbb{R})$: $\|f_2\|_{p_2}^{p_2}= \qquad =\|v\|_q^q$

$f_3\in L^{p_3}(\mathbb{R})$: $\|f_3\|_{p_3}^{p_3}=\int\left(|u(x-y)|^p|v(y)|^q\right)^{1/r\cdot r}=\int|u(x-y)|^p|v(y)|^q\,dx = (u^p*v^q)\in L^1$

$$\Longrightarrow \quad |u * v(x)| \le \int_{\mathbb{R}^d} |u(x-y)\, v(y)|^1 \, dy \underset{\boxed{2}}{\overset{\boxed{1}}{\le}} \|u\|_p^{p/p_1} \, \|v\|_q^{q/q_1} \left( \int |u(x-y)|^p |v(y)|^q \, dy \right)^{1/r}$$

$$\Longrightarrow \quad |u * v(x)|^r \le \|u\|_p^{r-p} \, \|v\|_q^{r-q} \left( \int_{\mathbb{R}^d} |u(x-y)|^p |v(y)|^q \, dy \right)$$

$$\overset{\int_{\mathbb{R}} \cdots dx}{\Longrightarrow} \quad \|u * v\|_r^r \le \|u\|_p^{r-p} \, \|v\|_q^{r-q} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |u(x-y)|^p |v(y)|^q \, dy \right) dx$$

$$\overset{\boxed{4}}{=} \|u\|_p^r \cdot \|v\|_q^r .$$

$\boxed{4}$

$$\|u\|_p^p \, \|v\|_q^q = \left( \int_{\mathbb{R}^d} |u(x)|^p \, dx \right) \left( \int_{\mathbb{R}^d} |v(y)|^q \, dy \right) = \int_{\mathbb{R}^d} |u(x-y)|^p \, dx \cdot \int_{\mathbb{R}^d} |v(y)|^q \, dy$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x-y)|^p \, |v(y)|^q \, dy \, dx \qquad \square$$

Beh $1 = \frac{1}{p} + \frac{1}{q}$, $u \in L^p(\mathbb{R}^d)$, $v \in L^q(\mathbb{R}^d)$ $\Rightarrow \|u * v\|_\infty \leq \|u\|_p \|v\|_q$

Bew Wegen $1 = 1/p + 1/q$ gilt mit Hölder'scher Ungleichung: $|u * v(x)| \leq$

$\leq \int_{\mathbb{R}^d} |u(x-y) v(y)| \leq \|u\|_p \|v\|_q$ also $\|u * v\|_\infty = \sup_{x \in \mathbb{R}^d} |u * v(x)| \leq$

$\|u\|_p \|v\|_q$ $\square$