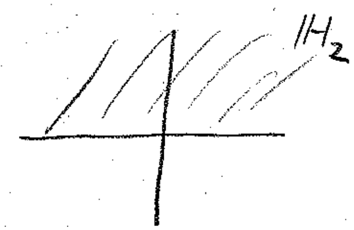
$\boxed{\text{Aufgabe 3}}$ $x' = (x, x_d) \in \mathbb{H} = \{ v \in \mathbb{R}^d \mid v_d > 0 \}$



$$P(x') = P(x, x_d) = \frac{2}{d\,\omega_d} \frac{x_d}{|(x, x_d)|^d} \qquad P: \mathbb{H} \to \mathbb{R}^+ \quad \text{stetig}$$

$$u(x') := u(x, x_d) = \big(P(\cdot, x_d) * g\big)(x) \qquad u: \mathbb{H} \to \mathbb{R}$$

$$= \int_{\mathbb{R}^{d-1}} P(x-y, x_d)\, g(x)\, dy$$

<u>Anwendung</u>: Jede harmonische Funktion lässt sich mit PK als Integral über ~~[illegible]~~ den Rand des Def.bereiches schreiben.

a) <u>Beh</u> $\int_{\mathbb{R}^{d-1}} P(x-y, x_d)\, dy = 1$ <u>Bew</u>: $\boxed{1}$ Zunächst ist das Integral aus $L^1([0,1])$ denn:

$$\int_{\mathbb{R}^{d-1}} \frac{x_d}{|(x-y, x_d)|^d}\, dy = \int_{\mathbb{R}^{d-1}} \frac{x_d}{(|x-y|^2 + x_d^2)^{d/2}}\, dy$$

$$= \int_{\mathbb{R}^{d-1}} \frac{x_d}{(|y-x|^2 + x_d^2)^{d/2}}\, dy$$

$$|(x-y, x_d)|^d = \sqrt{(x_1-y_1)^2 + \dots + x_d^2}^{\,d} = (|x-y|^2 + x_d^2)^{d/2}$$

$$\underset{z=y-x}{\overset{\uparrow}{=}} \int_{\mathbb{R}^{d-1}} \frac{x_d}{(|z|^2+x_d^2)^{d/2}}\, dz \underset{z := x_d \tau}{=} \int_{\mathbb{R}^{d-1}} \frac{x_d \cdot \overbrace{x_d^{d-1}}^{\det J\phi}}{(|x_d\tau|^2+x_d^2)^{d/2}}\, d\tau$$

$$\phi(\tau) := x_d \tau \qquad \phi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$$

$$\underset{x_d>0}{=} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tau|^2+1)^{d/2}}\, d\tau \quad \leftarrow f(|\tau|) \quad \underset{\text{Ana3}}{\overset{\text{Satz 2.38}}{=}} \underbrace{(d-1)\,\omega_{d-1}}_{<\infty} \int_0^\infty \frac{r^{d-2}}{(r^2+1)^{d/2}}\, dr \underset{\boxed{1}}{\overset{\boxed{2}}{<}} \infty$$

wegen $\boxed{1}$ $r \in [0,1]$: $\frac{r^{d-2}}{(r^2+1)^{d/2}} \le r^{d-2} \iff 1 \le (r^2+1)^{d/2}$ ✓ und

$r^{d-2} \in L^1([0,1])$ $\quad\boxed{2}$ $r \in [1,\infty]$: $\frac{r^{d-2}}{(r^2+1)^{d/2}} \le r^{-2} \iff r^d \le (r^2+1)^{d/2}$

$\iff r^2 \le r^2+1$ und $r^{-2} \in L^1([1,\infty])$

$\boxed{2}$

$$\int_{\mathbb{R}^{d-1}} P(x-y,\ x_d)\, dy \overset{\boxed{1}}{=} \frac{2}{d\,\omega_d} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tau|^2+1)^{d/2}}\, d\tau$$

$$= \frac{2}{d\,\omega_d} \cdot \overbrace{\frac{2}{\pi} \cdot \int_0^\infty \frac{s}{1+s^2}\, ds}^{=1} \cdot \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tau|^2+1)^{d/2}}\, d\tau$$

$\uparrow$

$$\int_0^\infty \frac{1}{1+s^2}\, ds = \arctan s\Big|_0^\infty = \frac{\pi}{2}$$

$$= \frac{4}{d\omega_d \pi} \int_0^\infty \int_{\mathbb{R}^{d-1}} \frac{1}{1+s^2} \cdot \frac{1}{(|\tau|^2+1)^{d/2}}\, d\tau\, ds$$

$$\overset{\boxed{1}}{=} \frac{4}{d\omega_d \pi} \int_0^\infty \int_{\mathbb{R}^{d-1}} \frac{1}{1+s^2} \cdot \frac{s}{(|z|^2+s^2)^{d/2}}\, dz\, ds =$$

$$= \frac{4}{d\omega_d \pi} \int_{\mathbb{H}} \frac{s}{(1+s^2)\,(|z|^2+s^2)^{d/2}}\, d(z,s)$$

CO-Area

$$= \frac{4}{\pi} \int_0^\infty r^{d-1} \int_{S^+} \frac{(r\xi_d)}{\underbrace{(1+(r\xi_d)^2)\left[\left((r\xi_1)^2+\dots+(r\xi_d)^2\right)\right]^{d/2}}_{= r^d \underbrace{|\xi|^d}_{=1} (1+(r\xi_d)^2)}} \, d\sigma(\xi)\, dr$$

$\uparrow$

$y = (z, s)$

$$= \frac{4}{\pi} \int_0^\infty r^{d-1} \int_{S^+} \frac{r\xi_d}{r^d (1+(r\xi_d))^2} \, d\sigma(\xi)\, dr$$

$$= \frac{4}{\pi} \int_0^\infty \int_{S^+} \frac{\xi_d}{1+(r\xi_d)^2} \, d\sigma(\xi)\, dr = \frac{4}{\pi} \int_{S^+} \int_0^\infty \frac{\xi_d}{(1+(r\xi_d)^2} \, dr\, d\sigma(\xi)$$

$$= \frac{4}{\pi} \int_{S^+} \frac{\pi}{2} \, d\sigma(\xi) = \frac{4}{2} \overbrace{\int_{S^+} d\sigma(\xi)}^{=1/2} = 1$$