$\boxed{\text{Aufgabe 1}}$ Bch: $T = \mathcal{L}(C[0,1])$ <u>Bew:</u>

$\boxed{1}$ $\int_0^1 \frac{1}{|s-t|^\alpha}\, dt \underset{\substack{z = s-t \\ dz = -dt}}{=} \int_s^{s-1} -\frac{1}{|z|^\alpha}\, dz$

$= \int_{s-1}^{s} \frac{1}{|z|^\alpha}\, dz \overset{s\in[0,1]}{=} \int_{s-1}^{0} \frac{1}{(-z)^\alpha}\, dz + \int_0^s \frac{1}{z^\alpha}\, dz$

$= -\frac{1}{1-\alpha} (-z)^{\overbrace{1-\alpha}^{>0}} \Big|_{s-1}^{0} + \frac{1}{1-\alpha} z^{\overbrace{1-\alpha}^{>0}} \Big|_0^s = \frac{1}{1-\alpha}(1-s)^{1-\alpha} + \frac{1}{1-\alpha} s^{1-\alpha} < \infty$

$\boxed{2}$ T stetig: $|Tx(s)| = \left| \int_0^1 \frac{x(t)}{|s-t|^\alpha}\, dt \right| \overset{x\in C([0,1])}{\leq} \|x\|_\infty \int_0^1 \frac{1}{|s-t|^\alpha}\, dt \overset{\boxed{1}}{<} \infty$

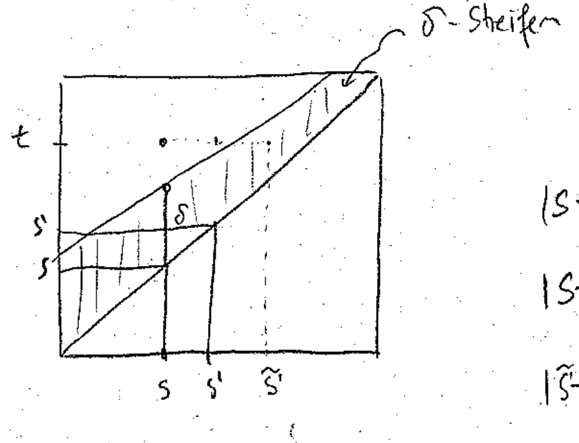
$\boxed{3}$ $Tx \in C([0,1])$: Wir zeigen die Stetigkeit von Tx, also sei $x \in C([0,1])$ vorgelegt. Wir zeigen $\forall \varepsilon > 0\ \exists \delta' > 0 : |s-s'| < \delta' \Rightarrow |Tx(s) - Tx(s')| < \varepsilon$ f. alle $s, s' \in [0,1]$. Es gilt $\boxed{3.1}$ $|Tx(s) - Tx(s')| =$

$\left| \int_0^1 x(t)\big(k(s,t) - k(s',t)\big)\, dt \right| \leq \|x\|_\infty \int_0^1 |k(s,t) - k(s',t)|\, dt = \|x\|_\infty \int_0^1 \left| \frac{1}{|s-t|^\alpha} - \frac{1}{|s'-t|^\alpha} \right| dt$. Problem ist die Diagonale $s = t$.

$\boxed{3.2}$ $\displaystyle\int_{|t-s|\geq\delta} |k(s,t) - k(s',t)|\,dt + \int_{|t-s|<\delta} |k(s,t) - k(s',t)|\,dt$ für ein ~~...~~ $\delta > 0$.

$\left(\delta < \frac{2}{3}\left(\frac{1}{4}(1-\alpha)\varepsilon\right)^{\frac{1}{1\alpha}}\right)$

$\boxed{3.2}$



$|s-t| \geq \delta$

$|s-s'| < \delta/2$

$|\tilde{s}-t| < \delta$

aber: $|s-s'| < \delta/2 \Rightarrow |t-s'| = |t-s+s-s'| \overset{\triangle}{\geq} |t-s| - |s-s'| \geq \delta - \delta/2 = \delta/2$

Daher: Aufg: $M = \{ (t,s) \in [0,1]^2 \mid |t-s| \geq \delta/2 \}$ ist k glm. stetig.

Zu $\varepsilon > 0$ gibt es also $0 < \delta' \overset{\text{o.E.}}{<} \delta/2$ so dass $|s-s'| = |\overbrace{(s,t)}^{\in M} - \overbrace{(s',t)}^{\in M}|$

$< \delta' \Rightarrow |k(s,t) - k(s',t)| < \varepsilon$. Für $|s-s'| < \delta'$ folgt also

$\displaystyle\int_{|t-s|\geq\delta} |k(s,t) - k(s',t)|\,d < \varepsilon \int_0^1 dt = \varepsilon$ denn mit $|t-s| \geq \delta \wedge |s-s'| < \delta'$

ist $|t-s'| \geq \delta - \delta' > \delta - \delta/2 = \delta/2$, also $(t,s), (t,s') \in M$

$\boxed{3.3.}$ In der Nähe der Diagonale gilt: $|t-s| < \delta$ und $|s-s'| < \delta'$

$$\Rightarrow \quad |t-s'| \leq |t-s| + |s-s'| < \delta + \delta' \overset{\boxed{3.2}}{<} \delta + \delta/2 = 3/2\,\delta$$

$$\Rightarrow \quad \{t \mid |t-s| < \delta,\ |s-s'| < \delta'\} \subset \{t \mid |t-s'| < 3/2\,\delta,\ |s-s'| < \delta'\}$$

$$\Rightarrow \quad \int_{|t-s|<\delta} |k(s,t) - k(s',t)|\, dt \leq \int_{|t-s|<\delta} |k(s,t)|\, dt + \int_{|t-s|<\delta} |k(s',t)|\, dt$$

$$\leq \int_{|t-s| \leq 3/2\,\delta} \frac{1}{|t-s|^\alpha}\, dt + \int_{|t-s'| < 3/2\,\delta} \frac{1}{|t-s'|^\alpha}\, dt = 2 \int_{|t-s| < 3/2\,\delta} \frac{dt}{|t-s|^\alpha} = 2 \int_{|z| < 3/2\,\delta} \frac{dz}{|z|^\alpha}$$

$z = t-s$

$dz = dt$

$$\leq 4 \int_0^{3/2\,\delta} \frac{1}{z^\alpha}\, dz = 4 \cdot \frac{1}{-\alpha+1}\, z^{\overbrace{-\alpha+1}^{>0}} \Big|_0^{3/2\,\delta} = \frac{4}{1-\alpha} \left(\tfrac{3}{2}\delta\right)^{1-\alpha} \overset{!}{<} \varepsilon$$

also $\left(\tfrac{3}{2}\delta\right)^{\alpha-1} < \frac{1}{4}(1-\alpha)\varepsilon \iff \delta < \frac{2}{3}\left(\frac{1}{4}(1-\alpha)\varepsilon\right)^{\frac{1}{\alpha-1}}$