$\boxed{6.4.a}$ $\boxed{1}$ $|a_k| = \dfrac{\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot (k-\frac{3}{2})}{k!} = \dfrac{1 \cdot 3 \cdot 5 \cdot \ldots (2k-3)}{2^k \cdot k!} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-1}{2k} \cdot \frac{1}{2k-1}$

$\Uparrow$ denn: $2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k = 2^k k!$

$\underline{k \mapsto k+1}$ $\quad 2^{k+1}(k+1)! = 2^k k! \cdot 2 \cdot (k+1) = 2 \cdot 4 \cdot 6 \cdots 2k \cdot 2(k+1)$

$\underline{k=1}:$ $2 = 2$ $\square$ $\Downarrow$

$= \frac{1}{a_k} \frac{1}{2(k-1)} = \frac{1}{\sqrt{k p_k}} \cdot \frac{1}{2k-1} \xrightarrow{p_k \to p \in [\sqrt{2}, 2]} \frac{1}{\sqrt{k p}} \frac{1}{2k-1} \leq \frac{1}{\sqrt{k}\sqrt{2}} \frac{1}{2k-1} \leq \frac{1}{\sqrt{k} \cdot k}$

$\boxed{2}$ Sei $f_k(x) = a_k x^k$. Dann $\sum\limits_{k \in \mathbb{N}_0} \sup\limits_{x \in [-1,1]} |f_k(x)| \leq \sum\limits_{k \in \mathbb{N}_0} |a_k| \leq \sum\limits_{k \in \mathbb{N}_0} \frac{1}{\sqrt{k} \cdot k}$

$< \infty$, denn $\sqrt[k]{\frac{1}{k\sqrt{k}}} = \frac{1}{\sqrt[k]{k}\,\sqrt[k]{\sqrt{k}}} \xrightarrow{k \to \infty} \underline{1} = g$ Mit Satz 9.5 (Ana 1)

folgt $g : [-1,1] \to \mathbb{R}$, $g(x) := \sum\limits_{k \in \mathbb{N}_0} f_k(x)$ glm stetig

und db auf $(-1,1)$.

$\boxed{6.4.b}$ $\underline{k=0}$ $a_1 = \frac{1}{2} a_0$ ✓ $\underline{k \geq 1}$ $(k+1)\, a_{k+1} + k\, a_k = (k+1) \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-(k+1)+1)}{(k+1)!}$

$$+ k \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} = \frac{1}{k!}\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\overbrace{\tfrac{1}{2}-(k+1)+1}^{=\frac{1}{2}-k}\right) + k\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\underbrace{\tfrac{1}{2}-k+1}_{\frac{1}{2}-(k-1)}\right)\right)\right)$$

$$= \underbrace{\frac{1}{k!}\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right)\right)}_{=a_k}\left(k + \left(\frac{1}{2}-k\right)\right) = \frac{1}{2} a_k$$

$\boxed{6.4.c}$

$$(1+x)\, g'(x) = (1+x) \sum_{k=1}^{\infty} k x^{k-1} a_k = \sum_{k=1}^{\infty} k x^{k-1} a_k + \sum_{k=1}^{\infty} k x^k a_k$$

$$= a_1 + \sum_{k=2}^{\infty} k x^{k-1} a_k + \sum_{k=1}^{\infty} k x^k a_k = a_1 + \sum_{k=1}^{\infty} (k+1) x^k a_{k+1} + \sum_{k=1}^{\infty} k x^k a_k$$

$$= a_1 + \sum_{k\in\mathbb{N}} x^k \left((k+1) a_{k+1} + k a_k\right) \overset{\boxed{b}}{=} \underbrace{a_1}_{=\frac{1}{2}a_0} + \frac{1}{2} \sum_{k\in\mathbb{N}} a_k x^k \; \cancel{\in \ldots g(x)}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} a_k x^k = \frac{1}{2} g(x)$$

$\boxed{6.4.d}$

$$h'(x) = \frac{1}{1+x} \cdot \left(g'(x)\sqrt{1+x} - \frac{1}{2\sqrt{1+x}} \cdot g(x) \right) = \frac{1}{1+x}\left(\overbrace{\frac{g(x)\sqrt{1+x}}{2(1+x)} - \frac{g(x)}{2\sqrt{1+x}}}^{=0} \right) = 0$$

~~Anfangsbedingung~~ $\Rightarrow g(x) = C \cdot \sqrt{1+x} \overset{h(0)=1=g(0)}{\Longrightarrow} g(x) = \sqrt{1+x}$

Also $g(x) \overset{(1)}{=} \sqrt{1+x}$ wobei

$\overset{(2)}{=} \sum_{k=0}^{\infty} a_k x^k$

in (1) $x \in (-1,1)$ und in (2) $x \in [-1,1]$. Zu zeigen ist

$g(1) = \sum a_k \overset{!}{=} \sqrt{2}$. Sei $h(x) := \sqrt{1+x}$, $h : (-1,1) \to \mathbb{R}$. Stetige

Fortsetzung von h: $x_n \to 1$ $h(1) := h(\lim x_n) = \sqrt{1+1} = \lim h(x_n)$

$= \lim g(x_n) \overset{g\ \text{stetig}}{=} g(\lim x_n) = g(1)$. Genauso bei -1.