Aufgabe 3

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - EX_1\right| > \varepsilon\right) \overset{T}{\leq} \frac{1}{\varepsilon^2} \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{\varepsilon^2 n^2} \operatorname{var}\left(\sum_{i=1}^{n} X_i\right)$$

$$\overset{①②③}{\leq} \frac{1}{\varepsilon^2 n^2}\left(n \cdot VX_1 + 2\sum_{j=1}^{n-1} r(j)\,|n-j|\right)$$

$$= \frac{VX_1}{\varepsilon^2 n} + \frac{2}{\varepsilon^2}\frac{1}{n^2}\sum_{j=1}^{n-1} r(j)(n-j)$$

$$= \frac{VX_1}{\varepsilon^2 n} + \frac{2}{\varepsilon^2}\sum_{j=1}^{n-1} \frac{r(j)n - j\,r(j)}{n^2}$$

$$= \frac{VX_1}{\varepsilon^2 n} + \frac{2}{\varepsilon^2}\sum_{j=1}^{n-1} r(j)/n\left[1 - \frac{j}{n}\right]$$

$$\xrightarrow{n\to\infty} 0 + \frac{2}{\varepsilon^2}\cdot 0 = 0$$

① $V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} VX_i + 2 \sum_{j=2}^{n} \sum_{i=1}^{j-1} cov(X_i, X_j)$

Bew (durch Induktion):

$n=2$: $V(X_1+X_2) = VX_1 + VX_2 + 2\,cov(X_1, X_2)$

$= \sum_{i=1}^{2} VX_i + 2\,(cov(X_1, X_2))$ ✓

$n \mapsto n+1$ $\quad V(\sum_{i=1}^{n+1} X_i) = V(\sum_{i=1}^{n} X_i + X_{n+1})$

$= V(\sum_{i=1}^{n} X_i) + VX_{n+1} + 2\,cov(\sum_{i=1}^{n} X_i, X_{n+1})$

$\overset{IV}{\underset{(*)}{=}} \sum^{n} VX + 2 \sum_{j=2}^{n} \sum_{i=1}^{j-1} cov(X_i X_j) + VX_{n+1}$

$+ 2 \sum_{i=1}^{n} cov(X_i, X_{n+1}) = \sum_{i=1}^{n+1} VX +$

$2 \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} cov(X_i X_j)$ ✓

(*) $cov(X+Y, Z) = E(X+Y)Z - E(X+Y)E(Z)$

$= EXZ + EYZ - EXEZ - EYEZ$

$= cov(X, Z) + cov(Y, Z)$

② $$\sum_{j=2}^{n} \sum_{i=1}^{j-1} \tau(|i-j|) = \underbrace{\tau(1)}_{j=2} + \underbrace{\tau(1) + \tau(2)}_{j=3}$$

$$+ \underbrace{\tau(1) + \tau(2) + \tau(3)}_{j=4} + \dots =$$

$$= (n-1)\,\tau(1) + (n-2)\,\tau(2) + \dots + 1 \cdot \tau(n-1)$$

$$= \sum_{j=1}^{n-1} \tau(j)\,(n-j)$$

③ $$\sum_{j=2}^{n} \sum_{i=1}^{j-1} \operatorname{cov}(X_i, X_j) \leq \sum_{j=2}^{n} \sum_{i=1}^{j-1} \tau(|i-j|)$$

$X_1, \dots, X_n$ pw. unkorreliert $\Rightarrow$ $\mathrm{cov}(X_i, X_j) = 0 \quad (i \neq j)$ $\quad X_i$

zz. $\quad \mathrm{var}\left(\sum^{n} X_i\right) = \sum^{n} \mathrm{var}\, X_i$

① $\quad \mathrm{cov}\left(\sum^{n-1} X_i, X_n\right) = E\left(\left(\sum X_i\right) \cdot X_n\right) - E\left(\sum X_i\right) E X_n$

$= E\left(\sum X_i X_n\right) - \sum E(X_i)\, E X_n$

$= \sum E X_i X_n - \sum E(X_i) E X_n = \sum \left(\overbrace{E X_i X_n - E X_i E X_n}^{= \mathrm{cov}(X_i, X_n) = 0}\right)$

$= 0$

② $\quad \mathrm{var}\left(\sum^{n} X_i\right) = \mathrm{var}\left(\sum^{n-1} X_i + X_n\right) \overset{①}{=} \mathrm{var} \sum^{n-1} X_i + \mathrm{var}\, X_n$

$\overset{\text{IVor}}{=} \sum^{n-1} \mathrm{var}\, X_i + \mathrm{var}\, X_n = \sum^{n} \mathrm{var}\, X_i$