$\boxed{\text{Aufgabe 3:}}$

<u>Beh</u> $P_{X_n} \xrightarrow{w} P_X$, $P_{U_n} \xrightarrow{w} \delta_u \Rightarrow P_{X_n+U_n} \xrightarrow{w} P_{X+u}$

<u>Bew</u> Es sei $f : \mathbb{R} \to \mathbb{R}$ glm. stetig. Zu $\varepsilon > 0$ wähle $\delta > 0$ so, dass $|f(x) - f(y)| < \varepsilon$ f. alle $|x-y| < \delta$. Dann rechne:

$$\left| \int_{\mathbb{R}} f \, dP_{U_n+X_n} - \int_{\mathbb{R}} f \, dP_{u+X} \right| = \left| \int_\Omega f(U_n+X_n) \, dP - \int_\Omega f(u+X) \, dP \right|$$

$$= \left| \int_\Omega f(u+X_n) - f(u+X) - f(u+X_n) + f(U_n+X_n) \, dP \right|$$

$=0$

$$\leq \left| \int_\Omega f(u+X_n) - f(u+X) \, dP \right| + \int_\Omega | f(U_n+X_n) - f(u+X_n) | \, dP$$

$$= \left| \int_\Omega f_u(X_n) - f_u(X) \, dP \right| + \int_{\{|U_n-u| \geq \delta\}} | f(U_n+X_n) - f(u+X_n) | \, dP + \int_{\{|U_n-u| < \delta\}} | f(U_n+X_n) - f(u+X_n) | \, dP$$

$\boxed{f_u(X_n) := f(u+X_n)}$

Es gelten: $\boxed{1}$ $\hat{f}_n \in C_b\mathbb{R}$ $\quad\boxed{2}$ $|f(U_n+X_n) - f(u+X_n)| \leq 2\|f\|_\infty < \infty$

$\boxed{3}$ $x := U_n + X_n,\ y := u + X_n \Rightarrow |x-y| = |U_n + X_n - u - X_n| = |U_n - u|$

Also obige Gleichung

$$\leq \underbrace{\left| \int_\Omega \hat{f}_n \, dP_{X_n} - \int_\Omega \tilde{f}_n \, dP_X \right|}_{\substack{P_{X_n} \xrightarrow{w} P_X \\ \downarrow \boxed{1} \\ 0}} + \underbrace{\underbrace{2\|f\|_\infty \int_{\{|U_n-u| \geq \delta\}} dP}_{\boxed{2}}}_{= 2\|f\|_\infty P(\{|U_n-u| \geq \delta\}) \xrightarrow{(1)} 0} + \underbrace{\varepsilon}_{\boxed{3}}$$

Insgesamt hat man also

$$\lim_{n\to\infty} \int_\Omega f \, dP_{U_n+X_n} = \int_\Omega f \, dP_{X+u}$$ für alle glm. stetigen f gezeigt.

Also $P_{U_n+X_n} \xrightarrow[w]{} P_{X+u}$ mit Portmanteau Thm.