Aufgabe 2 ii)

Beh: $\left(E_{n_K}(r_{n_K})\right)_{K\in\mathbb{N}}$ unabhängig

Bew: $n_{K+1} := n_K + \lceil r_{n_K} \rceil$, $n_1 := 1$, $n_2 = 1 + \lceil r_{n_1} \rceil = 1 + \lceil r_1 \rceil$, $n_3 = 1 + \lceil r_1 \rceil + \lceil r_{n_2} \rceil = 1 + \lceil r_1 \rceil + \lceil r_{1+r_1} \rceil$

$$\underbrace{1 + \lceil r_{n_1} \rceil + \lceil r_{n_2} \rceil}_{=n_3} + \underbrace{\lceil r_{n_3} \rceil}_{=n_4} + \lceil r_{n_4} \rceil + \lceil r_{n_5} \rceil = n_6$$

$$\{n_1, \dots, n_1 + \lceil r_{n_1} \rceil - 1\} = \overbrace{\{1, 2, \dots, \lceil r_{n_1} \rceil\}}^{=:D_1}, \quad \{n_2, \dots, n_2 + \lceil r_{n_2} \rceil - 1\} = \overbrace{\{1 + \lceil r_{n_1} \rceil, \dots, \lceil r_{n_1} \rceil + \lceil r_{n_2} \rceil\}}^{=:D_2}$$

$$\{n_3, \dots, n_3 + \lceil r_{n_3} \rceil - 1\} = \{1 + \lceil r_{n_1} \rceil + \lceil r_{n_2} \rceil, \dots, \lceil r_{n_1} \rceil + \lceil r_{n_2} \rceil + \lceil r_{n_3} \rceil\} =: D_3$$

$$\Rightarrow \mathbb{N} = \dot{\bigcup_{i=1}^{\infty}} D_i, \quad D_K := \{n_K, \dots, n_K + \lceil r_{n_K} \rceil - 1\}$$

$$\mathcal{B}_K := \bigcup_{i\in D_K} \{X_i = 1\}, \quad D_i \cap D_j \overset{i \neq j}{=} \emptyset$$

$$C_K = \{\omega \mid X_{n_K}(\omega) = \dots = X_{n_K + \lceil r_{n_K} \rceil - 1}(\omega) = 1\} = \{\omega \mid X_i = 1, i \in D_K\}$$

$$= \bigcap_{i\in D_K} \{X_i = 1\} \in \sigma(\mathcal{B}_K) = \sigma\Big(\underbrace{\bigcup_{i\in D_K} \{X_i = 1\}}\Big)$$

Satz 2.1.2

$\Longrightarrow \left(\sigma(\mathcal{B}_K)\right)_{K\in\mathbb{N}}$ unabhängig $\Rightarrow (C_K)_{K\in\mathbb{N}}$ unabh. , unabhängig $\forall i$ und $\cap$-stabil f. jedes i

Beh: $\sum_{k=1}^{\infty} P(E_{m_k}(r_{m_k})) = \sum_{k=1}^{\infty} 2^{-\lceil r_{m_k}\rceil} = \infty$

Bew: $\boxed{1}$ $r_{m_k} \leq \lceil r_{m_k} \rceil \overset{\circledast}{\leq} r_{m_k} + 1$

$$\Rightarrow \quad 2^{-\lceil r_{m_k}\rceil} \geq 2^{-r_{m_k}-1} = 2^{-r_{m_k}} 2^{-1}$$

$\boxed{2}$
$$2^{-r_{m_k}} = \frac{\lceil r_{m_k}\rceil}{\lceil r_{m_k}\rceil} \cdot 2^{-r_{m_k}} = \frac{m_{k+1} - m_k}{\lceil r_{m_k}\rceil} 2^{-r_{m_k}} = \frac{1}{\lceil r_{m_k}\rceil} \sum_{n=m_k}^{m_{k+1}-1} 2^{-r_{m_k}}$$

$$\geq \frac{1}{r_{m_k}+1} \sum_{n=m_k}^{m_{k+1}-1} 2^{-r_n} \geq \sum_{n=m_k}^{m_{k+1}-1} \frac{1}{r_n+1} 2^{-r_n}$$

$\circledast$
und $m_k \leq n \Rightarrow r_{m_k} \leq r_n$

$$\Rightarrow \sum_{k=1}^{\infty} 2^{-r_{m_k}} = \sum_{k=1}^{\infty} \sum_{n=m_k}^{m_{k+1}-1} \frac{1}{r_n+1} 2^{-r_n} = \sum_{n=1}^{\infty} \frac{2^{-r_n}}{r_n+1} \overset{\text{Vor.}}{=} \infty$$

$\square$

Also folgt $\sum_{k=1}^{\infty} P(E_{m_k}(r_{m_k})) = \infty \overset{BC}{\Longrightarrow} \overline{\lim_k} (E_{m_k}(r_{m_k})) = 1$

Beh $\quad \overline{\lim_{k\to\infty}} E_{n_k}(r_{n_k}) \subset \overline{\lim_{n\to\infty}} E_n(r_n)$

Bew $\quad \omega \underset{(*)}{\in} \bigcap_k \bigcup_{n_m \geq k} \{R_{n_m} \geq r_{n_m}\}$. Sei $s \in \mathbb{N}$. Dann

gibt es $k \in \mathbb{N}$: $n_m \geq s \;\; \forall m \geq k$ (denn $n_m \nearrow$ in m). Zu

diesem $k \in \mathbb{N}$ gibt es wegen $(*)$ $m \geq k$: $R_{n_m}(\omega) \geq r_{n_m}$.

Also gibt es $n_m \geq s$ mit $R_{n_m}(\omega) \geq r_{n_m}$, daher

$$\omega \in \bigcap_s \bigcup_{m \geq s} \{R_m \geq r_m\} = \overline{\lim_{s\to\infty}} \{R_s \geq r_s\}$$

$\square$

Also $P(\overline{\lim_{n\to\infty}} E_n(r_n)) \geq P(\overline{\lim} E_{n_k}(r_{n_k})) = 1$

$\boxed{\text{Aufgabe 2 iii}}$ $\quad r_n = \log_2(n)$.

$$x = 2^y \quad dx = \ln 2 \cdot 2^y \, dy$$

$$\sum_{n=1}^{\infty} \frac{2^{-\log_2(n)}}{\log_2(n)+1} = \sum_{n=1}^{\infty} \frac{1}{n(\log_2 n + 1)} \geq \int_{x \geq 1} \frac{1}{x(\log_2(x)+1)} \, dx = \int_0^{\infty} \frac{\ln 2 \, 2^y \, dy}{2^y(y+1)}$$

$$= \ln 2 \int_0^{\infty} \frac{1}{y+1} \, dy = \infty$$

$$\Longrightarrow P(\overline{\lim} \, E_n(r_n)) = 1 = P(\overline{\lim} \, E_n(\log_2(n))) = P(\overline{\lim} \, \{R_n \geq \log_2 n\}) = P\left(\overline{\lim} \frac{R_n}{\log_2 n} \geq 1\right)$$

Also: $P\left(\overline{\lim} \frac{R_n}{\log_2 n} = 1\right) = 1 - P\left(\overline{\lim} \frac{R_n}{\log_2 n} \neq 1\right)$

$$= \underbrace{1 - P(\overline{\lim} \ldots < 1)}_{=1} - \underbrace{P(\overline{\lim} \ldots > 1)}_{\substack{=0 \\ \text{(Blatt 3 A2)}}} = 1$$

$\square$