$\boxed{\text{Aufgabe 3}}$ Beh X intbar $\Leftrightarrow \sum_{n=1}^{\infty} P(|X| \geq \varepsilon n) < \infty$ (1)

Bew: Vorbetrachtungen: OE sei $X \geq 0$.

$\boxed{1}$ $A_n := \{\varepsilon n \leq X < \varepsilon(n+1)\} \Rightarrow \forall i \neq j: A_i \cap A_j = \emptyset$

$\boxed{2}$ $\varepsilon n\, P(A_n) = \int_{A_n} \varepsilon n \, dP \overset{\boxed{1}}{\leq} \int_{A_n} X \, dP \overset{\boxed{1}}{\leq} \varepsilon(n+1)\, P(A_n)$

$\boxed{3}$ $\sum_{n=1}^{N} \varepsilon n\, P(A_n) \overset{\boxed{2}}{\leq} \sum_{n=1}^{N} \int_{A_n} X \, dP \overset{\boxed{2}}{\leq} \sum_{n=1}^{N} \varepsilon(n+1)\, P(A_n)$

$\boxed{4}$ $B_n := \{X \geq \varepsilon n\}$ $\boxed{4a}$ $0 \leq \varepsilon N\, P(B_{N+1}) \leq \varepsilon(N+1)\, P(B_{N+1})$

$= \varepsilon(N+1) \int_{B_{N+1}} 1 \, dP \leq \int_{B_{N+1}} X \, dP$

$\boxed{4b}$ $\sum_{n=1}^{N} n\, P(A_n) + N\, P(B_{n+1}) = \sum_{n=1}^{N} P(B_n)$

Bew der Beh :

„$\Leftarrow$" Nach Vor. gilt $\infty > \sum_{n=1}^{\infty} P(X \geq n\varepsilon) = \sum_{n=1}^{\infty} P(B_n)$

$\overset{\boxed{4b}}{=} \lim_{N\to\infty} \sum_{n=1}^{N} n P(A_n) + N P(B_{N+1})$, also $\sum_{n=1}^{\infty} n P(A_n) < \infty$.

Daher $E(X) \overset{\boxed{1}}{=} \lim_{N\to\infty} \sum_{n=1}^{N} \int_{A_n} X \, dP \overset{\boxed{3}}{\leq} \lim_{N\to\infty} \sum_{n=1}^{N} \varepsilon(n+1) P(A_n)$

$= \lim_{N\to\infty} \varepsilon \sum_{n=1}^{N} P(A_n) + \varepsilon \sum_{n=1}^{N} n P(A_n) \overset{\sigma\text{-Ad}}{=} 1 + \varepsilon \sum_{n=1}^{\infty} n P(A_n) < \infty$

„$\Rightarrow$" Aus $\sum_{n=1}^{\infty} n P(A_n) \overset{\boxed{3}}{\leq} \sum_{n=1}^{\infty} \int_{A_n} X \, dP = E(X) < \infty$

und $\lim_{N\to\infty} \varepsilon N P(B_{N+1}) \overset{\boxed{4a}}{=} 0$ wg $B_N \downarrow \emptyset$ folgt

$\sum_{n=1}^{\infty} P(X \geq n\varepsilon) = \lim_{N\to\infty} \sum_{n=1}^{N} P(B_n) \overset{\boxed{4b}}{<} \infty$

Beweis $\boxed{4}\boxed{b}$: Es gilt $B_n \downarrow \emptyset$, $B_n \setminus B_{n+1} =$

$\{X \geq \varepsilon n\} \setminus \{X \geq \varepsilon(n+1)\} = A_n$.

$$\Rightarrow \sum_{n=1}^{N} n\, P(A_n) = \sum_{n=1}^{N} n\, P(B_n) - \sum_{n=1}^{N} n\, P(B_{n+1})$$

$$= \sum_{n=1}^{N} n\, P(B_n) - \sum_{n=1}^{N-1} n\, P(B_{n+1}) - N P(B_{N+1})$$

$$= \sum_{n=1}^{N} (n-1)\, P(B_n)$$

$$= \sum_{n=1}^{N} P(B_n) - N P(B_{N+1})$$