Aufgabe 2

$$g_{\alpha p}(x) = 1_{[0,\infty)}(x) \frac{1}{\Gamma(p)} x^{p-1} \alpha^{p} e^{-\alpha x}$$

$$(g_{\alpha p} * g_{\alpha q})(y) = \int_{-\infty}^{\infty} g_{\alpha p}(x)\, g_{\alpha q}(y-x)\, dx = \int 1_{[0,\infty)}(x)\, 1_{[0,\infty)}(y-x)$$

$$\cdot \frac{x^{p-1}(y-x)^{q-1}}{\Gamma(p)\Gamma(q)} \alpha^{p+q} e^{-\alpha x} e^{-\alpha(y-x)}\, dx = \frac{\alpha^{p+q} e^{-\alpha y}}{\Gamma(p)\Gamma(q)} \int 1_{[0,\infty)}(x)\, 1_{[0,\infty)}(y-x)\, x^{p-1}(y-x)^{q-1}\, dx$$

$$1_{[0,\infty)}(x) \cdot 1_{[0,\infty)}(y-x) = 1_{[0,\infty)}(x) \cdot 1_{(-\infty,y]}(x) = 1_{[0,\infty)\cap(-\infty,y]}(x)$$

$$= 1_{[0,y]}(x)\, 1_{[0,\infty)}(y)$$

$$z := \frac{x}{y} \quad dz = \frac{1}{y} dx$$

$$= \underbrace{\frac{\alpha^{p+q} e^{-\alpha y}}{\Gamma(p)\Gamma(q)} 1_{[0,\infty)}(y)}_{=:C} \int_0^y x^{p-1}(y-x)^{q-1}\, dx = C \int_0^1 (zy)^{p-1}(y-zy)^{q-1}\, y\, dz$$

$$= C \cdot y^{p+q-1} \int_0^1 z^{p-1}(1-z)^{q-1}\, dz = g_{\alpha, p+q}(y)$$

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Aufgabe 2 Schluss ohne Betafunktion:

Ausgerechnet: $$(g_{\alpha p} * g_{\alpha q})(y) \overset{\boxed{1}}{=} \frac{\alpha^{p+q} e^{-\alpha y} y^{p+q-1}}{\Gamma(p)\Gamma(q)} \int_0^1 z^{p-1}(1-z)^{q-1}\,dz$$

Wir wissen: $$\int_{-\infty}^{\infty} (g_{\alpha p} * g_{\alpha q})\,dy \underset{\boxed{2}}{=} 1$$

$$\int_{-\infty}^{\infty} g_{\alpha, p+q}(y)\,dy \underset{\boxed{3}}{=} 1$$

$$\overset{\boxed{1}}{\underset{\boxed{2}}{\Longrightarrow}} \quad 1 = \frac{1}{\Gamma(p)\Gamma(q)} \int_0^1 z^{p-1}(1-z)^{q-1}\,dz \cdot \int_{-\infty}^{\infty} \alpha^{p+q} e^{-\alpha y} y^{p+q-1}\,dy$$

$$\Longrightarrow \quad 1 = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 z^{p-1}(1-z)^{q-1}\,dz \cdot \underbrace{\int_{-\infty}^{\infty} \frac{\alpha^{p+q} e^{-\alpha y} y^{p+q-1}}{\Gamma(p+q)}\,dy}_{\overset{\boxed{3}}{=} 1}$$

$$\Longrightarrow \quad \int_0^1 z^{p-1}(1-z)^{q-1}\,dz = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\underset{\text{oben in Ausgerechnet einsetzen}}{\Longrightarrow} \quad g_{\alpha p} * g_{\alpha q} = g_{\alpha, p+q}$$