Aufgabe 3

i) $\hat{\mu}(0) = \int_{\mathbb{R}^n} e^0 \, d\mu = \int_{\mathbb{R}^n} 1 \, d\mu = \underline{1}$

$|\hat{\mu}(u)| \leq \int_{\mathbb{R}^n} |e^{iuy}| \, d\mu \leq \int_{\mathbb{R}^n} 1 \, d\mu \leq \underline{1}$

$\hat{\mu}(-u) = \int_{\mathbb{R}^n} e^{-iuy} \, d\mu = \int_{\mathbb{R}^n} \overline{e^{iuy}} \, d\mu = \overline{\int_{\mathbb{R}^n} e^{iuy} \, d\mu} = \overline{\hat{\mu}(u)}$

ii) Sei $\varepsilon > 0$. Zu $k \in \mathbb{R}$ sei $K := [-k,k]^n \subset \mathbb{R}^n$ und $\alpha := \sup_{y \in K} |y| < \infty$. Sei $\delta := \frac{\varepsilon}{\alpha}$ und sei k so groß, dass $\mu(\mathbb{R}^n \setminus K) < \varepsilon$. Zunächst gilt

$\boxed{1}$ $x \in \mathbb{R}$: $|1 - e^{ix}|^2 = |1 - \cos x - i \sin x|^2 = (1-\cos x)^2 + \sin^2 x$

$= 1 - 2\cos x + 1 = 2 - 2\cos x = 4 \cdot \frac{1}{2}(1 - \cos x) \overset{\frac{1}{2}\text{Winkel}}{\leq} 4 \cdot (\sin \frac{x}{2})^2 \leq 4 \cdot (\frac{x}{2})^2 = x^2$

$\boxed{2}$ $|e^{iuy} - e^{iu'y}| = |e^{iuy} \cdot (1 - e^{iu'y} e^{-iuy})| = |1 - e^{i(u'y - uy)}|$

$= |1 - e^{i(u'-u)y}| \overset{\boxed{1}}{\leq} |(u'-u)y| \overset{\text{CSU}}{\leq} |u'-u||y| \quad \left(\overset{y \in K}{\leq} |u'-u| \alpha\right)$

Sei nun $|u'-u| \le \delta$. Dann

$\boxed{3}$ $$|\hat{\mu}(u) - \hat{\mu}(u')| \le \int_{\mathbb{R}^n} |e^{iuy} - e^{iu'y}| \, d\mu = \int_{\mathbb{R}^n \setminus K} \overbrace{|e^{iuy} - e^{iu'y}|}^{\le |e^{iuy}| + |e^{iu'y}| = 2} \, d\mu$$

$$+ \int_K |e^{iuy} - e^{iu'y}| \, d\mu \underset{\boxed{2}}{\le} 2\mu(\mathbb{R}^n \setminus K) + |u-u'| \alpha \cdot \mu(K)$$

$$\le 2\varepsilon + \alpha \delta \mu(K) = 2\varepsilon + \alpha \frac{\varepsilon}{\alpha} \mu(K) = \varepsilon(2 + \mu(K))$$

$\square$

iii) Es sei $z := \sum_{j=1}^{m} c_j e^{iu_j y}$. Dann gilt

$$0 \le |z|^2 = z\bar{z} = \left(\sum_{j=1}^{m} c_j e^{iu_j y}\right)\left(\overline{\sum_{j=1}^{m} c_j e^{iu_j y}}\right) = \sum_{k,j=1}^{m} c_j e^{iu_j y}\, \overline{c_k e^{iu_k y}}$$

Damit: $$\sum_{j,k=1}^{m} c_j \bar{c}_k \hat{\mu}(u_j - u_k) = \sum_{j,k=1}^{m} c_j \bar{c}_k \int_{\mathbb{R}^m} e^{i(u_j - u_k)y} \, d\mu$$

$$= \int_{\mathbb{R}^m} \sum_{k,j=1}^{m} c_j e^{iu_j y} \bar{c}_k e^{-iu_k y} \, d\mu = \int_{\mathbb{R}^m} \sum_{k,j=1}^{m} c_j e^{iu_j y}\, \overline{c_k e^{iu_k y}} \ge 0$$