Aufgabe 4

2.5.8 : [redacted] $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iyu} e^{-\frac{1}{2}|y|^2} \, dy$ [redacted] $= e^{-\frac{1}{2}|u|^2}$

i) $\varphi_X(u) = E(e^{iuX}) = \int_{\mathbb{R}} e^{iuX} \, dP = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{iux} \, e^{\frac{|x-u|^2}{2\sigma^2}} \, dx$

$$\underset{\substack{\uparrow \\ z := \frac{x-m}{\sigma}, \; dz = \frac{1}{\sigma} dx}}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{iu(\sigma z + m)} \, e^{\frac{1}{2}|z|^2} \, \sigma \, dz = \frac{1}{\sqrt{2\pi\sigma^2}} \sigma e^{ium} \int_{\mathbb{R}} e^{i(u\sigma)z} \, e^{\frac{1}{2}|z|^2} \, dz$$

$$= e^{ium} \, e^{-\frac{1}{2}|u\sigma|^2} = e^{ium} \, e^{-\frac{1}{2}|u|^2 \sigma^2}$$

ii) $\varphi_{X+Y}(u) = \varphi_X(u) \, \varphi_Y(u) = e^{ium_1} e^{-\frac{1}{2}|u|^2 \sigma_1^2} \, e^{ium_2} e^{-\frac{1}{2}|u|^2 \sigma_2^2}$

$$= e^{iu(m_1+m_2)} \, e^{-\frac{1}{2}|u|^2(\sigma_1^2+\sigma_2^2)} = \widehat{N(m_1+m_2, \sigma_1^2+\sigma_2^2)}$$

$$= \widehat{P}_{X+Y} \overset{\text{Eind.satz}}{\Longrightarrow} N(m_1+m_2, \sigma_1^2+\sigma_2^2) = P_{X+Y}$$