Stable pseudofinite groups

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Abstract

The main theorem is that if G is a pseudofinite group with stable theory, then G has a definable normal soluble subgroup of finite index.

1 Introduction

In this paper, we shall say that a group G is *pseudofinite* if it is an infinite model of the theory of finite groups; that is, if it is elementarily equivalent to an infinite ultraproduct of finite groups. The purpose of the paper is to prove the following.

Theorem 1.1 *Let Γ be a pseudofinite group with stable theory. Then Γ has a definable soluble normal subgroup of finite index.*

The key step in the proof of Theorem 1.1 is the following proposition, from which the theorem follows easily (using the classification of finite simple groups). The proposition ensures that any pseudofinite stable group has a largest soluble normal subgroup (which will be definable). The existence of this is an open question for stable groups in general. It seems that our methods make essential use of pseudofiniteness, so do not answer the general question.

Proposition 1.2 *Let Γ be a pseudofinite group with stable theory. Then there is $k \in \mathbb{N}$ such that every soluble normal subgroup of Γ has derived length at most k.*

There is not a lot of literature on pseudofinite groups. The main result is the theorem of J.S. Wilson [19], that any infinite pseudofinite *simple* group is elementarily equivalent to a Chevalley group (possibly of twisted type) over a pseudofinite field. In fact, Point [9] shows that an ultraproduct of simple Chevalley groups (of fixed type) over finite fields is isomorphic to the Chevalley group over the ultraproduct of the fields and that a similar result holds for the twisted case. In addition (Proposition 3 of [9]) she verifies simplicity of the ultraproduct, in both the untwisted and the twisted cases.

There is some discussion at the beginning of Section 5 of a number of other results, due to Khelif and others, which were communicated to us by G. Sabbagh. Part of our interest comes from recent work of Macpherson and Steinhorn, extended

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by that of Elwes and Ryten, on *asymptotic classes* of finite structures. These are classes of finite structures which satisfy the good asymptotic results on definable sets that hold for finite fields (see [2]). It has been shown that any ultraproduct of members of a 1-dimensional asymptotic class of groups is finite-by-abelian-by-finite, and (by Elwes and Ryten) any ultraproduct of members of a 2-dimensional asymptotic class of groups is soluble-by-finite. Furthermore, Elwes has shown that any stable ultraproduct of an asymptotic class is 1-based.

In Section 2, we list some background results used later. Then in Section 3 we prove the technical result Proposition 3.1. This immediately yields the proof of Theorem 1.1 once we have Proposition 1.2, and is also used in the proof of Proposition 1.2. In Section 4 we give a proof of Proposition 1.2. Section 5 contains a short discussion of some related work, and an example.

2 Background results

We recall some well-known facts about stable groups, which can all be found in [17]. First, by the Baldwin-Saxl Theorem, in a stable theory if G is a group acting definably on a definable set X, and $Y \subset X$, then there is a finite subset $F \subset Y$ such that $G_{(Y)}$ (the pointwise stabiliser in G of Y) is equal to $G_{(F)}$. More generally, G has *icc*, the uniform chain condition on intersections of uniformly definable subgroups – see Definition 1.0.3 of [17]): for any formula ϕ there is some $n_\phi < \omega$ such that any chain of intersections of ϕ-definable subgroups of G has length at most n_ϕ. In particular, there is a natural number n such that any chain of centralisers in G has length at most n.

Furthermore, we have the following, a mixture of results of Berline and Lascar [1] and Wagner [16] – see Theorems 1.1.10 and 1.1.12 of [17].

Fact 2.1 *Let G be a stable group, and $k \in \mathbb{N}$. Then*

(i) if H is a soluble subgroup of G, then there is a definable soluble supergroup of H of the same derived length lying in G, with the defining formula depending only on the derived length;

(ii) there is a definable soluble normal subgroup R_k of G such that every soluble normal subgroup of G of derived length at most k lies in R_k.

By Theorem 2.1(ii), if G is stable and there is an upper bound on the derived lengths of the soluble normal subgroups of G, then G has a unique largest soluble normal subgroup, and the latter is definable.

Since solubility of derived length at most d is determined by a group law, it is easily seen that if the stable group G has a definable soluble normal subgroup of finite index, then so does every group elementarily equivalent to G. In particular, in the proofs of Proposition 1.2 and Theorem 1.1, we shall assume that the group Γ of Theorem 1.1 is an ultraproduct $\Pi_\omega G_i/\mathcal{U}$ of finite groups G_i ($i \in \omega$), with respect to a non-principal ultrafilter $\mathcal{U}$ on ω.

In the proofs below, a non-principal ultrafilter $\mathcal{U}$ on ω is fixed, and we shall say that $I \subset \omega$ is *large* if $I \in \mathcal{U}$, and I is *small* if $\omega \setminus I \in \mathcal{U}$. In the proofs of Theorem 1.1 and Proposition 1.2, we argue by contradiction, working with a supposed counterexample Γ presented as an ultraproduct $\Gamma = \Pi_\omega G_i/\mathcal{U}$. Sometimes, for some property P of groups, we identify a large set $I \subset \omega$ such that for $i \in I$, all the G_i have property P. In this case, $\mathcal{U}$ induces an ultrafilter $\mathcal{U}_I$ on I, and Γ is elementarily equivalent to $\Gamma_I := \Pi_I G_i/\mathcal{U}_I$. Hence the latter will also be a counterexample. Thus, in such a situation we may replace ω by I and Γ by Γ_I, but to avoid proliferation of notation, we keep the symbols $\mathcal{U}$, Γ, and may re-use the symbol I later.

In Section 3 we use the following facts. The notion of 'twisted group' over an appropriate pseudofinite field makes sense – it is essentially the same construction as over a finite field.

Fact 2.2 *Let G be a simple Chevalley group, possibly of twisted type, over a pseudofinite field F. Then*
(i) the field F is definable (with parameters) in G;
(ii) the theory of G is unstable.

Proof. (i) An explicit and uniform first-order formula defining the underlying field with two parameters was given in [7] for (untwisted) Chevalley groups (and one can check that the same formula works for the twisted groups as well, see e.g. [15] 7.6.7 and 7.7.19). The field is obtained by defining a maximal torus T as the center of the centralizer of an element t of T and considering the orbit of this group on a central element u of a unipotent subgroup.

(ii) This follows from (i), since by Duret [5], the theory of any pseudofinite field is unstable.

Remark. Results like that of (i) appear to be folklore. The interpretability of the field was proved by S. Thomas in the unpublished PhD thesis [14]. A similar result is proved by M. Ryten in [11]. He shows that, except in the case of Suzuki and Ree groups, the group is bi-interpretable (over parameters) with the field. In the Suzuki and Ree case, it is bi-interpretable over parameters with an expansion of the field by a certain automorphism.

Lemma 2.3 *(i) There is a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that if G is a finite nilpotent group of class at most 2 generated by a subset of size k, and H is a soluble group of automorphisms of G whose Fitting subgroup has class at most ℓ, then H has derived length at most $f(k, \ell)$.*

(ii) Let N be a nilpotent class 2 group with cyclic centre Z, let $d \in \mathbb{N}$, and suppose that no subset of N/Z of size d generates N/Z. Then N has a strict descending chain of centralisers of length $d + 1$.

Proof. (i) Since G is nilpotent, it is a direct product of (at most k) characteristic Sylow p-subgroups, so we may assume that G is a p-group. Let $\Phi(G)$ be the Frattini subgroup of G, and put $\bar{G} := G/\Phi(G)$. Then $\bar{G}$ is an elementary abelian p-group. By the Burnside Basis Theorem (5.3.2 of [10]), the dimension of $\bar{G}$ as a vector space over F_p is k, where we assume k is the size of a minimal set of generators for G. Let $\psi : \mathrm{Aut}(G) \to \mathrm{Aut}(\bar{G})$ be the natural map, and let $H \leq \mathrm{Aut}(G)$ be a solvable subgroup. Then $\mathrm{Ker}(\psi)$ is nilpotent, as it is a p-group (5.3.3. of [10]), so $\mathrm{Ker}(\psi) \cap H$ has nilpotency class at most ℓ. On the other hand, by a theorem of Zassenhaus ([20], see Theorem 16 of [13]), there is a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that the derived length of any soluble subgroup of $\mathrm{Aut}(\bar{G})$ is at most $\lambda(k)$. Thus, H has derived length at most $\lambda(k) + \ell$.

(ii) For each $x \in N \setminus Z$ there is a homomorphism $\phi_x : N/Z \to Z$ given by $\phi_x(gZ) = [g, x]$. Then $(N/Z)/\ker(\phi_x)$ is cyclic, and as $C_N(x)$ is the preimage of $\ker(\phi_x)$ under the map $N \to N/Z$, $N/C_N(x)$ is cyclic. There is a finite set $\{x_1, \ldots, x_r\}$, chosen of minimal possible size, such that $Z = C_N(x_1, \ldots x_r)$. Then $N > C_N(x_1) > \ldots > C_N(x_1, \ldots, x_r) = Z$. Now N/Z is a subdirect product of r cyclic groups $N/C_N(x_1), \ldots, N/C_N(x_r)$, so is generated by r elements. Hence, $r > d$.

Corollary 2.4 *Let d be a positive integer, and for each $i \in \omega$ let N_i be a finite d-generator nilpotent group of class at most 2, and H_i be a group of automorphisms of N_i. Suppose that some infinite ultraproduct $H = \Pi_\omega H_i/\mathcal{U}$ is stable. Then there* *is a large set $J \subset \omega$ and positive integer c such that for each $j \in J$, the soluble radical $R(H_j)$ of H_j has derived length at most c.*

Proof. First, observe that if $R = \Pi_\omega R(H_i)$, then R is a subgroup of the stable group H, so is an $\mathcal{M}_c$-group, that is, has the descending chain condition on centralisers. It follows from [4] (see also Theorem 1.2.11 of [17]), that the Fitting subgroup of R (the group generated by the nilpotent normal subgroups of R) is nilpotent. Hence, there is some e and a large $J \subset \omega$ such that for all $j \in J$ the Fitting subgroup of $R(H_j)$ has class at most e. The result now follows from Lemma 2.3(i).

3 Bounded groups and proof of Theorem 1.1

In this section, we prove the following proposition which will be used in the proof of Proposition 1.2. Theorem 1.1 then follows immediately from the proposition and Proposition 1.2.

For any group G, if G has a unique largest soluble normal subgroup, this is called the *radical* of G, and denoted by $R(G)$. For convenience, we shall say that a group is *unbounded* if for each $d \in \mathbb{N}$ it has a soluble normal subgroup of derived length greater than d; otherwise, G is *bounded*.

Proposition 3.1 *Let Γ be a pseudofinite group with stable theory. Then Γ is bounded if and only if Γ has a definable soluble characteristic subgroup of finite index.*

Proof. If Γ has a definable soluble characteristic subgroup of finite index, then clearly Γ is bounded.

For the other direction we suppose that $\Gamma := \Pi_\omega G_i/\mathcal{U}$ is stable. As Γ is bounded, Γ has a unique largest soluble normal subgroup $R(\Gamma)$, and $R(\Gamma)$ is definable, of derived length k, say, and characteristic in Γ. There is a first order sentence which expresses of a group G that G has no non-trivial abelian normal subgroup: $(\forall x \neq 1)(\exists y)([x, x^y] \neq 1)$. Thus, the fact that $\Gamma/R(\Gamma)$ has no non-trivial abelian normal subgroup is expressible, as is the fact that $R(\Gamma)$ has derived length k. Hence, replacing ω by a large subset if necessary, we may suppose that for each $i \in \omega$, the soluble radical $R(G_i)$ of G_i is definable, uniformly in i, and has derived length k. For each $i \in \omega$, let N_i be the socle of $G_i/R(G_i)$, so N_i is a direct product $N_i = S_{i,1} \times \ldots \times S_{i,r_i}$ of non-abelian simple groups. Let $\bar{S}_{i,j}$, $\bar{N}_i$ be the preimages of $S_{i,j}$, N_i respectively under the map $G_i \to G_i/R(G_i)$.

Claim 1. There is large $I \subset \omega$ and $r \in \mathbb{N}$ such that for all $i \in I$, $r_i \leq r$.

Proof. If the claim is false then for all $r \in \mathbb{N}$ there is large $I_r \subset \omega$ such that $r_i > r$ for $i \in I_r$. Now for $i \in I_r$, and $j = 1, \ldots r$, choose $x_{ij} \in S_{i,j} \setminus \{1\}$. Then $C_{N_i}(x_{i,1}), C_{N_i}(x_{i,1}, x_{i,2}), \ldots, C_{N_i}(x_{i,1}, x_{i,2}, \ldots, x_{i,r})$ is a proper descending chain of centralisers. It follows that $\Gamma/R(\Gamma)$ has a proper descending chain of centralisers of length r. As r is arbitrarily large and $R(\Gamma)$ is definable, this contradicts stability.

By Claim 1, replacing ω by a large subset if necessary, we may suppose that $r_i = r$ for all $i \in \omega$.

Claim 2. There is large $I \subset \omega$ and $s \in \mathbb{N}$ such that for each $i \in I$ and $j = 1, \ldots, r$, if $S_{i,j}$ is an alternating group A_n then $n \leq s$, and if $S_{i,j}$ is a Chevalley group then its Lie rank is at most s.

Proof. Suppose that for arbitrarily large s there is large $J_s \subset \omega$ such that $S_{i,1}$ is the alternating group A_s. The group A_s has a centraliser chain of length at least the integer part of $s/2$ (consider centralisers of disjoint transpositions), so under this assumption Γ has an infinite descending chain of centralisers, a contradiction.

Similarly, the groups $S_{i,1}$ cannot be of fixed Chevalley type and arbitrarily large rank: to see this consider centralizers of central elements in the unipotent radical of the minimal parabolics. If in the associated building, the minimal parabolics move along a minimal gallery of maximal length, we obtain a proper descending chain of centralizers (see [18], 11.11, 11.17). Since the maximal length of a minimal gallery increases with the rank, we obtain a contradiction.

Since there are finitely many types of Chevalley group, this suffices.

To prove the proposition, we must show that there is $e \in \mathbb{N}$ and large $J \subset \omega$ such that for $i \in J$, $|N_i| \leq e$. This then gives an upper bound e' on $|\operatorname{Aut}(N_i)|$ (for $i \in J$), and hence on $|G_i/R(G_i)|$, since $G_i/R(G_i)$ embeds in $\operatorname{Aut}(N_i)$. In particular, $|\Gamma/R(\Gamma)| \leq e'$, as required.

Arguing by contradiction, we may suppose that for each $e \in \mathbb{N}$ there is large $J_e \subset \omega$ such that for $i \in J_e$, $|N_i| > e$. By Claims 1 and 2, using again that there are finitely many Chevalley types, we may suppose that there is a fixed type of finite simple group $\operatorname{Chev}(q)$, of fixed Lie rank m, and for each $n \in \mathbb{N}$ there is large $K_n \subset \omega$ such that for each $i \in K_n$, $S_{i,1}$ is isomorphic to $\operatorname{Chev}(q)$ with $q > n, m$.

There is a natural number d and for each i some $g_i \in S_{i,1}$ such that each element of $S_{i,1}$ is a product of at most d conjugates (in $S_{i,1}$) of g_i and g_i^{-1}. (In fact, by the result of Point mentioned in the introduction, the ultraproduct of the $S_{i,1}$ will itself be simple, so for suitable d *any* non-identity g_i will do.) It follows that $S_{i,1}$ is uniformly definable in G_i, as the set of all elements which are a product of at most d $Z_{i,1}$-conjugates of g_i and g_i^{-1}. It follows, again by the result of Point mentioned in the introduction, that a group elementarily equivalent to a Chevalley group over a pseudofinite field is interpretable in Γ. By Fact 2.2, it follows that Γ has unstable theory, a contradiction.

Remark 3.2 Let G be a group. Then G is unbounded if and only if for every definable normal subgroup N of G, either N or G/N is unbounded.

For the right-to-left condition, suppose that G is bounded. Then putting $N = \{1\}$, both N and G/N are bounded.

For the left-to-right direction, suppose that T is a soluble normal subgroup of G of derived length at least $2d$, and N is an arbitrary normal subgroup of G. Then either $T \cap N$ is a soluble normal subgroup of N of derived length at least d, or TN/N is a soluble normal subgroup of G/N of derived length at least d.

This observation is used repeatedly, sometimes without explicit reference. We thank Martin Ziegler for pointing out this easy argument.

4 Proof of Proposition 1.2

In this section we prove Proposition 1.2, and hence Theorem 1.1 (in view of Proposition 3.1). We begin with two lemmas.

Lemma 4.1 *Suppose that G has a soluble normal subgroup R of derived length e and index f, and let N, S be normal subgroups of G with $N < S$ and S/N soluble. Then the derived length of S/N is at most $e + f$.*

Proof. Replacing R by $R \cap S$, we may suppose that $R \leq S$. Now $N \leq RN \leq S$, and RN/N is isomorphic to $R/R \cap N$ so has derived length at most e, and S/RN has order, and hence derived length, at most f.

Lemma 4.2 *Let G be an unbounded stable group. Then there is in G an interpretable abelian group A, and an unbounded interpretable group H, such that H has an interpretable faithful action on A as a group of automorphisms.*

Proof of Lemma 4.2. First, by Fact 2.1(ii), there is a function $g : \mathbb{N} \to \mathbb{N}$ and an increasing chain of definable normal subgroups $(R_i : i \in \omega)$ of G, with $R_0 := \{1\}$, such that for each i, R_i contains all soluble normal subgroups of G of derived length i, and is definably soluble of derived length $g(i)$. In particular, $g(i+1) \geq g(i)$, and $g(i) \to \infty$. Put $R := \bigcup(R_i : i \in \omega)$. By the chain condition on centralisers, there is $n \in \mathbb{N}$ such that $Z(R) = C_R(R_n)$. Thus, $R/Z(R) = R/(C_G(R_n) \cap R) \cong RC_G(R_n)/C_G(R_n)$, a normal subgroup of $G/C_G(R_n)$. Clearly $R/Z(R)$ is unbounded, so has characteristic soluble subgroups of arbitrarily large derived length. Hence $G/C_G(R_n)$ is also unbounded.

Put $m := g(n)$. There is a chain $S_0 < S_1 < \ldots < S_m = R_n$ of definable normal subgroups of R_n such that each S_{i+1}/S_i is abelian. Let ℓ be maximal such that $G/C_G(S_\ell)$ is bounded. Then the group $A := S_{\ell+1}/S_\ell$ is abelian. Also, $C_G(S_\ell)/C_G(S_{\ell+1})$ is unbounded by Remark 3.2. Let $N := C_{C_G(S_\ell)}(A)$. Then $C_G(S_{\ell+1}) \leq N \leq C_G(S_\ell)$, and $C_G(S_\ell)$ induces $C_G(S_\ell)/N$ on A. Since $N/C_G(S_{\ell+1})$ consists of automorphisms of $S_{\ell+1}$ which fix S_ℓ and $A = S_{\ell+1}/S_\ell$ pointwise, it embeds in a direct power of the soluble group S_ℓ, so is soluble. It follows that $C_G(S_\ell)/N$ is unbounded, that is, G induces on A an unbounded interpretable group of automorphisms.

Proof of Proposition 1.2. Let $\Gamma := \Pi_\omega G_i/\mathcal{U}$ be stable, and suppose for a contradiction that Γ is unbounded. By the last lemma, there is in Γ an interpretable abelian group A, an unbounded interpretable group H, and an interpretable faithful action of H on A. We shall write A additively. For each $x \in A$, put $x^H := \{x^h : h \in H\}$, where x^h denotes the image of x under h. Then $C_H(x^H) \trianglelefteq H$. We want to reduce to the situation where H induces an unbounded group on each x^H. Let $B := \{x \in A : H/C_H(x^H) \text{ is bounded}\}$. Clearly, B is H-invariant.

First notice that B is a group. For suppose $x, y \in B$. Then $H/C_H(x^H)$ and $H/C_H(y^H)$ are both bounded, so soluble-by-finite, and $H/(C_H(x^H) \cap C_H(y^H))$ is a subdirect product of $H/C_H(x^H)$ and $H/C_H(y^H)$ so is also soluble-by-finite and hence bounded. Now $C_H(x^H) \cap C_H(y^H) \leq C_H((xy)^H)$, so by Remark 3.2, $H/C_H((xy)^H)$ is also bounded.

In order to show that B is definable we need part (i) of the following:

Claim 1. (i) There is some $e \in \mathbb{N}$ such that for all $x \in B$, the radical of $H/C_H(x^H)$ has derived length at most e.

(ii) The radical of $H/C_H(B)$ has derived length at most e.

Proof of Claim 1. (i) By the descending chain condition on centralisers, there is finite $B_0 \subset B$ such that $C_H(B) = C_H(B_0)$. Let $B_0 = \{x_1, \ldots, x_r\}$, and let $H/C_H(x_i^H)$ have radical of derived length e_i, with $e_1 \leq \ldots \leq e_r$. Put $N := C_H(B_0)$. Then $N = C_H(x_1^H) \cap \ldots \cap C_H(x_r^H)$ (as $B_0 \subseteq x_1^H \cup \ldots \cup x_r^H \subseteq B$). Thus H/N is a subdirect product of the groups $H/C_H(x_1^H), \ldots, H/C_H(x_r^H)$, so has radical of derived length at most e_r. In particular, H/N is bounded, so by Proposition 3.1 its radical has finite index d, say. Now let $x \in B$. Then $C_H(x^H)$ is a normal subgroup of H containing N. Hence, by Lemma 4.1 applied to H/N, $H/C_H(x^H)$ has radical of derived length at most $e_r + f$.

(ii) In the proof of (i), we showed that $H/C_H(B)$ has radical of derived length at most e_r.

By Claim 1(i), B is definable, and by (ii), B is a *proper* subgroup of A: for H is unbounded and $C_H(A) = 1$.

We now reduce to the case when $B = \{0\}$. Let $x \in A \setminus B$. Then $H/C_H(x^H)$ acts faithfully on x^H, and preserves an equivalence relation $\sim$, where $u \sim v$ if $uB = vB$. Let F be the subgroup of H consisting of elements which fix each $\sim$-class of x^H. We claim that H/F (the group induced by H on the set of $\sim$-classes) is unbounded. We have $H/F \cong [H/(C_H(x^H) \cap F)]/[F/(C_H(x^H) \cap F)]$. Now for $x \notin B$, $H/(C_H(x^H) \cap F)$

is unbounded because $H/C_H(x^H)$ is unbounded. It is left to show that for $x \notin B$, $F/(C_H(x^H) \cap F)$ is bounded. By Claim 1 (i), we know that $H/C_H(B)$ is bounded, and hence $F/(C_H(B) \cap F)$ is bounded. Notice that for $h \in C_H(B) \cap F$ there is some $b \in B$ such that $x^h = x + b$, and thus $C_H(B) \cap F$ induces an abelian group on x^H. Putting this together, we see that F induces a bounded group on x^H (for $x \notin B$), and thus $F/(F \cap C_H(x^H))$ is bounded as required.

We have shown that H induces an unbounded group on $(xB)^H$. Thus, we may replace A by A/B; that is, we may assume that *for all* $x \in A \setminus \{0\}$, H induces an unbounded group on x^H.

At this point, it is convenient to work with the finite groups G_i. By Los's Theorem and the above definability, after first replacing ω by a large subset if necessary, we may suppose that for all i there are groups A_i and H_i interpretable in G_i, with H_i acting faithfully on A_i; furthermore these groups and their actions are *uniformly* interpretable, though possibly with parameters.

For each $i \in \omega$, choose a minimal H_i-invariant normal subgroup V_i of A_i. Then V_i is elementary abelian, i.e. a vector space over some prime field, and H_i acts linearly on it. We do not claim that the V_i are definable uniformly in i. However, by the descending chain condition on centralisers (applied in G), $C_{H_i}(V_i)$ is definable uniformly in i. Put $K_i := H_i/C_{H_i}(V_i)$. Then K_i is uniformly interpretable in G_i, and acts faithfully and irreducibly on V_i. If $x_i \in V_i \setminus \{0\}$, then $C_{H_i}(V_i) = C_{H_i}(x_i^{H_i})$, and so $K_i = H_i/C_{H_i}(x_i^{H_i})$. Thus, by the last paragraph but one, $\Pi_\omega K_i/\mathcal{U}$ is unbounded, so for each $d \in \mathbb{N}$ there is large $J_d \subset \omega$ such that for $i \in J_d$, K_i has radical of derived length at least d.

For each $i \in \omega$, let X_i be a maximal abelian normal subgroup of K_i. Since $X_i := C_{K_i}(C_{K_i}(X_i))$, it is definable, uniformly in i. Let $J := \{i \in \omega : X_i = Z(K_i)\}$.

Claim 2. J is small.

Proof of Claim 2. Suppose for a contradiction that J is large. In this case, we may suppose that $J = \omega$. By Schur's Lemma, X_i is cyclic, for all i. For each $i \in \omega$, let $T_i := \mathrm{Soc}(R(K_i)/X_i)$, and $\bar{T}_i$ be the preimage of T_i under the map $K_i \to K_i/X_i$. Then $\bar{T}_i$ is nilpotent of class at most 2. Put $Y_i := C_{K_i}(\bar{T}_i)$. Then $Y_i \cap \bar{T}_i$ is an abelian normal subgroup of K_i containing X_i, so by maximality of X_i, $Y_i \cap \bar{T}_i = X_i$. In particular, as $\mathrm{Soc}(R(Y_i)/X_i) \leq T_i$, $R(Y_i) \leq X_i$. Hence, for all $d \in \mathbb{N}$ and $i \in J_d$, K_i induces on $\bar{T}_i$ a group of automorphisms with radical of derived length at least $d - 1$.

It follows, by applying Corollary 2.4 to the set of pairs $\{(\bar{T}_i, K_i/C_{K_i}(\bar{T}_i)) : i \in \omega\}$, that for each $c \in \omega$ there is a large subset $L_c \subset \omega$ such that for $i \in L_c$, any generating subset of $\bar{T}_i$ has size at least c. Hence, by Lemma 2.3(ii), if $i \in L_c$ then $\bar{T}_i$ has a descending chain of centralisers of length c. It follows that if $\bar{T} := \Pi_\omega \bar{T}_i/\mathcal{U}$, then $\bar{T}$ has an infinite descending chain of centralisers. Since $\bar{T}$ is a subgroup of $K := \Pi_\omega K_i/\mathcal{U}$ which is interpretable in the stable group Γ, this is a contradiction, so proves the claim.

By the claim, we may assume that $J = \emptyset$, so for each $i \in \omega$, $Z(K_i)$ is a proper subgroup of X_i.

Claim 3. There is $n \in \mathbb{N}$ and a large set $J \subset \omega$ such that for $i \in J$, X_i is generated by n elements.

Proof of Claim 3. Suppose not. Then for all n, the set $\{i \in J : X_i \text{ is } n\text{-generated}\}$ is small, i.e., the set $L_n := \{i \in J : X_i \text{ is not } n\text{-generated}\}$ is large. We will show that if X_i is not n-generated, then X_i contains a properly descending sequence of centralizers of length at least n.

Let $i \in L_n$. Then we can construct a sequence of non-trivial subspaces $U_i^1, \ldots, U_i^n$ of V_i and a descending sequence $X_i > X_i^1 > \ldots > X_i^n$ in X_i as follows. Choose U_i^1 to be any non-trivial irreducible X_i-submodule of V_i, and put $X_i^1 := C_{X_i}(U_i^1)$.

By Schur's Lemma, X_i/X_i^1 is cyclic, so X_i^1 is not $(n-1)$-generated. If we have found $U_i^1, \ldots, U_i^r$ and $X_i^1 > \ldots > X_i^r$, choose U_i^{r+1} to be any non-trivial irreducible X_i^r-submodule of V_i, and $X_i^{r+1} := X_i^r/C_{X_i^r}(U_i^{r+1})$. Then, by induction, X_i^{r+1} is not $(n-r-1)$-generated. Now the sequence $C_{X_i}(U_i^1), \ldots, C_{X_i}(U_i^1 + \ldots + U_i^n)$ is a proper descending sequence of centralisers in X_i.

Since n is arbitrary, it follows that some group interpretable in Γ has an infinite descending chain of centralisers, a contradiction.

We now fix n as in Claim 3, and again, we may assume that $J = \omega$. That is, we assume that for all $i \in \omega$, X_i is n-generated.

Let $Y_i := C_{K_i}(X_i)$. It follows from Claim 3 and Corollary 2.4 (applied to the set $\{(X_i, K_i/Y_i) : i \in \omega\}$), that for some $n' \in \mathbb{N}$ and large $J \subset \omega$, if $i \in J$ then $R(K_i/Y_i)$ has derived length at most n'. Hence, $R(Y_i)$ has large derived length. More precisely, for each $m \in \mathbb{N}$ there is a large subset $L_m \subset J$ such that for $i \in L_m$, $R(Y_i)$ has derived length at least m: indeed, put $L_m := J \cap J_{m+n'}$ (where $J_{m+n'}$ is such that K_i has radical of derived length at least $m + n'$ for $i \in J_{m+n'}$.)

For each $i \in \omega$, by Clifford's Theorem there is a direct sum decomposition $V_i = V_i^1 \oplus \ldots \oplus V_i^{r_i}$ into irreducible non-trivial Y_i-modules, such that the (abstract) groups $Y_i/C_{Y_i}(V_i^j)$ are isomorphic (we do not claim that for fixed i the V_i^j are isomorphic Y_i-modules). In particular, let K_i^1 be the group induced by Y_i on V_i^1. Then Y_i is isomorphic to a subdirect power of K_i^1, so $R(K_i^1)$ has derived length at least as big as that of $R(Y_i)$.

We now iterate the above process with (K_i^1, V_i^1) replacing (K_i, V_i). For each i, we obtain a sequence $(K_i^0, V_i^0) = (K_i, V_i), (K_i^1, V_i^1), \ldots, (K_i^t, V_i^t), \ldots$ as follows: for each i, K_i^j acts irreducibly and faithfully on V_i^j, K_i^j has a maximal abelian normal subgroup X_i^j, and this is not central in K_i^j, $Y_i^j := C_{K_i^j}(X_i^j)$, V_i^{j+1} is a non-trivial irreducible Y_i^j-submodule of V_i^j, and Y_i^j induces K_i^{j+1} on V_i^{j+1}. For each j, the groups K_i^j will be uniformly (in i) interpretable in the G_i, as they arise by taking centralisers. We do not claim uniform interpretability of the V_i^j.

There are now two cases. Suppose first that for arbitrarily large $t \in \mathbb{N}$, there is large $I_t \subset \omega$ such that for $i \in I_t$, there are at least t values of j such that K_i^{j+1} is a proper quotient of Y_i^j. This means that the sequence $C_{K_i}(V_i), C_{K_i}(V_i^1), \ldots, C_{K_i}(V_i^j), \ldots$ contains at least t distinct groups. It follows that a group interpretable in Γ has an infinite chain of centralisers, a contradiction.

In the other case, for each t there is $r = r(t)$ and a large $I_t \subset \omega$ such that for each $i \in J_t$, and each $j = r, r+1, \ldots, r+t$, Y_i^j acts faithfully on V_i^{j+1}. Now Y_i^j is a proper subgroup of K_i^j, since $X_i^j \neq Z(K_i^j)$. It follows that we have a proper descending chain of centralisers $K_i^j > K_i^{j+1} = C_{K_i^j}(X_i^j) > K_i^{j+2} > \ldots > K_i^{j+t}$. Thus, a group interpretable in Γ has a descending chain of centralisers of length t for arbitrary t. This contradiction proves the lemma.

5 Further remarks on stable pseudofinite groups

When a first draft of this paper was written, we did not know whether every stable pseudofinite group is nilpotent-by-finite. However, we understand from G. Sabbagh that the answer to this is negative. Indeed, a certain group was constructed independently by Chapuis [3] and by Simonetta (Application 3.2.5 of [12]), and shown by them to be ω-stable, metabelian, but not nilpotent-by-finite. Then Khelif, again independently, constructed a group elementarily equivalent to this one from scratch, and showed it to be pseudofinite. He has shown that any pseudofinite ω-stable group of finite Morley rank is abelian-by-finite. Zilber, also, has sketched to us a construction of a pseudofinite superstable soluble group which is not nilpotent-by-finite. Essentially, one finds an appropriate small but infinite subgroup Γ of $(\mathbb{C}, \cdot)$, such that $(\mathbb{C}, +, \times, \Gamma)$ is superstable, and shows that the semidirect product $(\mathbb{C}, +).\Gamma$ is pseudofinite. Zilber's construction may give a group elementarily equivalent to the Chapuis-Simonetta group, but the approach is different.

For general interest, we give below a construction of a pseudofinite ω-stable group which is nilpotent-by-finite but not abelian-by-finite. A different construction of such a group was also known to Sabbagh, based on earlier work of Rahantarijao.

Example 5.1 We construct an ω-stable pseudofinite group which is not abelian-by-finite. The construction uses Mekler's method for building nilpotent class 2 groups from graphs, preserving various model-theoretic properties [8]. We follow the explanation in Appendix A.3 of [6], using the notion of *special model* also described in [6].

Choose an infinite cardinal λ which is an uncountable *strong limit number* in the sense of [6], that is, a limit beth. Then any countably infinite structure over a countable language has a special elementary extension of cardinality λ (Theorem 10.4.2 of [6]).

For each n, let Δ_n be the (undirected) graph on $\{0, 1, \ldots, n-1\}$, where a and b are joined if $|a - b| = 1 \pmod{n}$. Also, let Δ be the graph on $\mathbb{Z}$ where a, b are joined if $|a-b| = 1$. Let $\Delta(\lambda)$ be the graph consisting of a disjoint union of λ many copies of Δ. Finally, let $\Delta(\lambda)^+$ be the graph consisting of $\Delta(\lambda)$ together with λ many isolated vertices. The graphs Δ_n (for $n \geq 5$), Δ, and the $\Delta(\lambda)$ are all *nice* in the sense of [6]. (A *nice graph* is one without squares or triangles, and such that for any distinct vertices a, b, there is a vertex c joined to a but not b.)

Fix an odd prime p, and let $\mathcal{N}_p$ be the variety of nilpotent groups of class 2 and exponent p. For each cardinal λ, let $V(\lambda)$ be an elementary abelian p-group of cardinality λ. Given a nice graph Γ, one can form a group $G(\Gamma)$ which is freely generated by the vertices of Γ, subject to the laws of $\mathcal{N}_p$ and the relation $[a, b] = 1$ where a, b are adjacent vertices of Γ. By Corollary A.3.11 of [6], the graph Γ is interpretable in $G(\Gamma)$ (uniformly in Γ). However, $G(\Gamma)$ is not in general interpretable in Γ. Hodges gives an incomplete theory T_{ng} which holds of all groups $G(\Gamma)$, where Γ is a nice graph.

The graph Δ (which is elementarily equivalent to the $\Delta(\lambda)$), is strongly minimal. It follows that $G(\Delta(\lambda))$ is ω-stable. (For this, see either Theorem 2.11 of [8], or Corollary A.3.19 of [6], whose proof does not use uncountability of λ.)

Furthermore, if H is a special model of cardinality λ of $\mathrm{Th}(G(\Delta(\lambda)))$, then $H \cong G(\Delta(\lambda)^+) \times V(\lambda)$, by A.3.15 of [6].

Suppose now that G is any infinite special model, of cardinality λ, of the theory of the groups $G(\Delta_n)$. Then $G \models T_{ng}$. Also, the graph $\Gamma(G)$ will be an infinite special model of the theory of the graphs Δ_n, so will be isomorphic to $\Delta(\lambda)$. It follows, again from Corollary A.3.15 of [6], that $G \cong G(\Delta(\lambda)^+) \times V(\lambda)$. In particular $G \cong H$, so H is a *pseudofinite* ω-stable group.

Finally, we check that the group $G(\Delta(\lambda))$ (and hence H) is not abelian-by-finite. So suppose for a contradiction that A is an abelian normal subgroup of $G(\Delta)$ of finite index. It is easily checked that at most 2 of the generating vertices (elements of $\Delta(\lambda)$) lie in A. Thus, there is a copy of Δ, say $\{a_i : i \in \mathbb{Z}\}$ in $\Delta(\lambda)$, such that none of the a_i lie in A. Since $|G(\Delta(\lambda)) : A|$ is finite, by the pigeon-hole principle there are $i < j < k$ in $\mathbb{Z}$ such that $a_i a_j^{-1}$ and $a_j a_k^{-1}$ lie in A. In particular, they commute, so $(a_j a_k^{-1})(a_i a_j^{-1}) = a_i a_k^{-1}$. The latter contradicts the uniqueness of support mentioned in the paragraph following the proof of Lemma A.3.2 in [6].

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