Gabriel-Roiter measures for the 3-Kronecker quiver

Philipp Fahr

Fakultät für Mathematik, Universität Bielefeld,
Postfach 100131, D-33501 Bielefeld, Germany
E-mail: philfahr@math.uni-bielefeld.de

Abstract

We will construct indecomposable representations of the 3-Kronecker quiver and obtain uncountably many infinite Gabriel-Roiter measures. Our aim is to classify all piling submodules of an indecomposable regular module. A possible largest Gabriel-Roiter measure in the central part is discussed.

Key words: Quiver, Gabriel-Roiter measure, extended Kronecker quiver.
Mathematics Subject Classification: Primary: 16G20, Secondary: 16G60, 16D90.

---

1 Introduction

We will investigate the Gabriel-Roiter measure of indecomposable regular representations of the 3-Kronecker quiver, having two vertices and three arrows in the same direction. We will construct uncountably many infinite Gabriel-Roiter measures, which is done with the help of *piling submodules*. We classify these piling submodules of a certain indecomposable regular module, and we will show that they are either unique of a certain length or there is a one-parameter family of such submodules.

While constructing infinite-dimensional representations of the 3-Kronecker quiver, their Gabriel-Roiter measures are determined. These will be infinite Gabriel-Roiter measures lying in the so-called *central part* (see [Ri06]). Contrary to the tame hereditary case, the wild 3-Kronecker quiver has uncountably many infinite Gabriel-Roiter measures.

Preprint submitted to Elsevier *5 May 2008*

Let A be a finite-dimensional algebra.

For the definition of the Gabriel-Roiter measure we need to define an ordering on the set of all subsets $\mathcal{P}$ of natural numbers. Consider the following relation on $\mathcal{P}$: let I, J be subsets of the natural numbers with $I \neq J$. Then $I < J$ provided the smallest element in the symmetric difference (i.e. in $(I \backslash J) \cup (J \backslash I)$) belongs to J.

Definition 1 *Let M be an A-module. Let $\mu(M)$ be the supremum (with the above total order) of the sets $\{|M_1|, \ldots, |M_t|\}$, where $M_1 \subset M_2 \subset \ldots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the* Gabriel-Roiter measure *of M.*

We call an inclusion $N \subset M$ of indecomposable A-modules a *Gabriel-Roiter inclusion*, if $\mu(M) = \mu(N) \cup \{|M|\}$. For a Gabriel-Roiter inclusion $N \subset M$, the module M/N is indecomposable. If M is not of finite length, let $\mu(M)$ be the supremum of the numbers $\mu(M')$ taken over all (indecomposable) submodules M' of M of finite length. As Gabriel-Roiter filtration one defines the following: Let M be a module which is not finitely generated. A sequence $M_1 \subset M_2 \subset \ldots \subset M_t \subset \ldots$ is called a *Gabriel-Roiter filtration* of M provided the following conditions are satisfied:

(i) M_1 is a simple module.
(ii) M_{i-1} is a Gabriel-Roiter submodule of M_i, for all $i \geq 2$.
(iii) $M = \bigcup_i M_i$.

The following properties of the Gabriel-Roiter measure are taken from [Ri06] and [Ri05], where full proofs can be found.

- For any non-zero module M, there is an indecomposable submodule $M' \subset M$ with $\mu(M') = \mu(M)$. Thus in the definition of the Gabriel-Roiter measure it suffices to consider only indecomposable proper submodules.
- For any module M, the Gabriel-Roiter measure $\mu(M)$ is the supremum of $\mu(M')$, where M' is a finitely generated indecomposable submodule of M.
- Let M be a module and $N \subset M$ a submodule. Then $\mu(N) \leq \mu(M)$. If M is indecomposable and N a proper submodule of M, then $\mu(N) < \mu(M)$.

An important theorem of C.M. Ringel is (see [Ri05]):

Theorem 2 *Any module M with a Gabriel-Roiter filtration is indecomposable.*

Let us define piling submodules.

Definition 3 *A submodule W of a module M is a* piling submodule*, provided either $W = 0$ or W is indecomposable and the Gabriel-Roiter measure of M starts with that of W, i.e. $\mu(W) = \mu(M) \cap \{1, 2, \ldots, |W|\}$.*

Piling submodules have the following nice properties, which are easily verified:

- The zero submodule of a module M is always a piling submodule, as is any submodule of M of length 1.
- Any submodule with simple socle of a module M is a piling submodule.
- Piling submodules of M of the same length have the same Gabriel-Roiter measure.
- If W is a piling submodule of M, then all submodules of W occurring in a Gabriel-Roiter filtration of W are piling submodules of M.

C.M. Ringel has conjectured that there are only countably many Gabriel-Roiter measures in the case of a tame algebra. There exists an unpublished result stating that in the tame hereditary case this is true. We will now consider the the wild hereditary 3-Kronecker quiver. We will construct uncountably many infinite Gabriel-Roiter measures.

1.2 The 3-Kronecker quiver

The 3-Kronecker quiver $K(3)$ is the following quiver: $$\begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha"'] \arrow[l, "\beta"] \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

From now on let $A = kK(3)$ be the path algebra of the quiver $K(3)$, where k is an algebraically closed field. A representation of $K(3)$ is of the form $(V_1, V_2; \alpha, \beta, \gamma)$, where V_1, V_2 are k-vector spaces and $\alpha, \beta, \gamma : V_2 \to V_1$ are three linear transformations. The path algebra A is a connected wild hereditary algebra, which is finite-dimensional and has basis $\{e_1, e_2, \alpha, \beta, \gamma\}$, where e_1, e_2 are the trivial paths at vertices $1, 2$ respectively.

For the quiver $K(3)$ we have two simple modules S_1, S_2 with dimension vectors $\mathbf{dim}\, S_1 = (1, 0)$ and $\mathbf{dim}\, S_2 = (0, 1)$. We also have two projective modules P_1, P_2 and two injective modules I_1, I_2 with the following dimension vectors respectively: $\mathbf{dim}\, P_1 = (1, 0)$, $\mathbf{dim}\, P_2 = (3, 1)$, and $\mathbf{dim}\, I_1 = (0, 1)$, $\mathbf{dim}\, I_2 = (1, 3)$.

1.3 Definition of $R[n]_I$ and Universal Covering

In section 2 we will be working with the representations $R[n]_I$ defined as follows: Let $n \in \mathbb{N}_1$ and I a subset of $\mathbb{N}_1$. Define the representation $R[n]_I$ for the quiver $K(3)$ as follows: the vector space $R[n]_I^{(1)}$ has basis z_i, $1 \leq i \leq n$, the vector space $R[n]_I^{(2)}$ has basis x_j, y_i, where $1 \leq j \leq n$, and $1 \leq i \leq n$ with $i \in I$. Let

$$\alpha(x_j) = z_j,\ \beta(x_j) = z_{j-1},\ \text{and}\ \gamma(x_j) = 0,$$
$$\alpha(y_i) = 0,\ \beta(y_i) = 0,\ \text{and}\ \gamma(y_i) = z_i,$$

for all $1 \leq i, j \leq n$, $i \in I$, with $z_0 = 0$.

If $I \cap \{1, \ldots, n\} = \emptyset$, then for any $n \in \mathbb{N}_1$, $R[n]_I$ equals its restriction to the maps α, β. In this case $R[n]_I$ can be viewed as a representation over the 2-Kronecker quiver, where $R[n]_I$ restricted to α, β is indecomposable. In general, let us show that the modules $R[n]_I$ are indecomposable for every $n \in \mathbb{N}_1$ and every set $I \subseteq \mathbb{N}_1$. This is done using covering techniques and we will make use of the 3-regular tree (3-regularity means that every vertex has precisely 3 neighbours). This has useful properties in connection with the 3-Kronecker quiver, since it is just the universal covering of the 3-Kronecker quiver.

In general, if the quiver Q is considered as the universal covering (even Galois covering) of the quiver $\overline{Q}$, then any representation V of Q gives rise to a representation $\overline{V}$ of $\overline{Q}$ (by attaching to the vertex 0 of $\overline{Q}$ the direct sum of the vector spaces attached to the various sinks of Q, and attaching to 1 the direct sum of the vector spaces attached to the sources). The quivers which are of interest in our case are, on the one hand, the 3-Kronecker quiver $\overline{Q}$ and, on the other hand, the bipartite quiver Q whose underlying graph (obtained by deleting the orientation of the arrows) is the 3-regular tree. The covering functor $V \mapsto \overline{V}$ preserves indecomposability and satisfies $\mathbf{dim}\,\overline{V} = \mathbf{dim}\,V$. Since in the case of $R[n]_I$ each vector space attached to the sinks and sources is one-dimensional and the quiver Q is connectred we get:

Proposition 4 *Let $n \in \mathbb{N}_1$, $I \subseteq \mathbb{N}_1$. Then $R[n]_I$ is indecomposable.*

Proposition 4 has been shown in [CB98], theorem 1.4, where the special case of tree modules has been given particular consideration.

2 Construction of uncountably many Gabriel-Roiter measures

Let $n \in \mathbb{N}_1$ and I a subset of $\mathbb{N}_1$. Consider the representation $R[n]_I$ as defined in 1.3. Then $R[n]_I$ only depends on $I \cap \{1, 2, \ldots, n\}$ and not on I itself, and we have inclusion maps $R[0]_I \subset R[1]_I \subset R[2]_I \subset \ldots \subset R[n]_I \subset \ldots$. Our aim is to prove the following theorem, classifying all piling submodules of $R[n]_I$ of length at least 3 in the case $1 \in I$.

Theorem 5 (Main Theorem) *Let $n \in \mathbb{N}_1$ and let $I \subset \mathbb{N}_1$ with $1 \in I$.*

(1) *For any $1 \leq m \leq n$, $R[m]_I$ is the unique piling submodule of length $|R[m]_I|$.*

(2) *For any* $1 \leq m \leq n$ *with* $m+1 \in I$, *and for every* $\mu \in k$, $R[m]_I + A(x_{m+1} + \mu y_{m+1})$ *is a piling submodules of length* $|R[m+1]_I| - 1$.
(3) *There are no other piling submodules of length at least* 3.

Remarks 1

(1) Note that $|R[m]_I| = 2m + |I \cap \{1, 2, \dots, m\}|$ *is a number depending only on* m *and* $I \cap \{1, 2, \dots, m\}$. *Thus the theorem states in particular the existence of piling submodules of certain lengths.*
(2) In the case $1 \in I$, *the modules in* (2) *form a one-parameter family of piling submodules of* $R[m+1]_I$, *which are maximal in* $R[m+1]_I$ *and contain* $R[m]_I$. *These piling submodules form an affine line.*
(3) The simple modules are the only piling submodules of length 1.
(4) The piling submodules of length 2 *are classified in proposition 9.*

Corollary 6 *Let* $n \in \mathbb{N}_1$ *and* $I \subset \mathbb{N}_1$ *with* $1 \in I$. *Then any piling submodule of* $R[n]_I$ *of length at least* 3 *belongs to a Gabriel-Roiter filtration of* $R[n]_I$.

Remarks 2

(1) There is only one submodule of length 1 *occurring in any Gabriel-Roiter filtration of* $R[n]_I$, *namely* $\operatorname{soc} R[1]_I$.
(2) The modules of length 2 *occurring in a Gabriel-Roiter filtration of* $R[n]_I$, $1 \in I$, *are as follows: There is a one-parameter family of the form* $A(x_1 + \mu y_1)$, *with* $\mu \in k$. *In addition we have a submodule of length* 2 *generated by* $\langle y_1 \rangle$. *Thus these piling submodules form a projective line.*
(3) By theorem 5 we know that $R[1]_I$, *when* $1 \in I$, *is the only piling submodule of* $R[n]_I$ *of length* 3. *By above remark we know all Gabriel-Roiter submodules and* $\mu(R[1]_I) = \{1, 2, 3\}$. *Therefore any Gabriel-Roiter measure of* $R[n]_I$ *starts with* $\mu(R[1]_I)$.
(4) Since $R[m]_I$ *is the unique piling submodule of* $R[n]_I$, *for* $m \leq n$, $1 \in I$, $R[m]_I$ *occurs in every Gabriel-Roiter filtration of* $R[n]_I$.

The proofs of above theorem and its corollary will be given in section 2.2. First we collect some preliminary results.

Proposition 7 *(1) If* W *is a piling submodule of* M *and there is an indecomposable module* W' *with* $W \subset W' \subseteq M$, *such that* $|W'| = |W| + 1$, *then* W' *is a piling submodule of* M.
(2) Let W *be a piling submodule of* M *and* W' *an indecomposable module, such that* $W \subset W' \subseteq M$ *and* $|W'| = |W| + 2$. *Assume further that if* X *is a piling submodule of* M *with* $|X| = |W|$, *then there is no indecomposable submodule* X' *with* $X \subset X'$ *and* $|X'| = |X| + 1$. *Then* W' *is a piling submodule of* M.

PROOF. The first result follows using the definition of the Gabriel-Roiter measure. For the proof of the second statement proceed by contradiction. Since W is piling, $\mu(W) = \mu(M) \cap \{1, \ldots, |W|\}$. Let $\mu(M) = \{l_1, l_2, \ldots, l_i, \ldots, l_r\}$, where $l_1 < l_2 < \ldots < l_i < \ldots < l_r$ and $l_i = |W|$. Then $\mu(W) = \{l_1, \ldots, l_i\}$. Now assume $l_{i+1} = l_i + 1$. There exists a chain of indecomposable modules $X_1 \subset X_2 \subset \ldots \subset X_{i+1}$ with $|X_j| = l_j$, $1 \leq j \leq i+1$. So $X = X_i$ is piling, $|X| = l_i = |W|$ and $X' = X_{i+1}$ is indecomposable with $|X'| = |X| + 1$. This, however, contradicts the assumption that there is no indecomposable X' with $X \subset X'$ and $|X'| = |X| + 1$. Therefore we must have $l_{i+1} \geq l_i + 2$. Since we have an indecomposable module W', such that $W \subset W' \subseteq M$ and $|W'| = |W| + 2 = l_i + 2$, we have $l_{i+1} = l_i + 2 = |W'|$. Thus $\mu(W') = \mu(W) \cup \{|W'|\} = \{l_1, \ldots, l_i, l_{i+1}\} = \mu(M) \cap \{1, \ldots, |W'|\}$, since W is piling. So W' is piling too.

For $0 \leq m \leq n$ let us construct a map $f : R[n]_I \to R[m]_I$, such that $\mathrm{Ker}(f)$ is $R[n-m]_J$, where the set $J = \{i - m \,|\, i \in I, i > m\}$. We get the following isomorphism:

Lemma 8 (General Structure Lemma) *Let* $n \in \mathbb{N}_1$, *and* $I \subset \mathbb{N}_1$. *Then* $R[n]_I / R[m]_I \cong R[n-m]_J$, $0 \leq m \leq n$, *and* $J = \{i - m \,|\, i \in I, i > m\}$.

PROOF. For the proof we make the following convention for the notation of the basis: for $i \leq 0$, $x_i = 0$, $y_i = 0$, and $z_i = 0$, where $R[n]_I^{(1)}$ has basis z_i, $1 \leq i \leq n$, and $R[n]_I^{(2)}$ has basis x_j, y_i, where $1 \leq j, i \leq n$, and $i \in I$. Then $f : R[n]_I \to R[n-m]_I$ will be defined on each vector space as follows:

$$f^{(2)}(x_j) = x_{j-m},\ f^{(2)}(y_i) = y_{i-m} \text{ in case } i \in I, \text{ and } f^{(1)}(z_i) = z_{i-m}.$$

We are left to check that f is a homomorphism: 1. $f^{(1)}\alpha = \alpha f^{(2)}$, 2. $f^{(1)}\beta = \beta f^{(2)}$ and 3. $f^{(1)}\gamma = \gamma f^{(2)}$, where $\alpha(x_j) = z_j, \beta(x_j) = z_{j-1}, \gamma(x_j) = 0$, and $\alpha(y_i) = 0, \beta(y_i) = 0, \gamma(y_i) = z_i$, for all $1 \leq i, j \leq n$, $i \in I$.

2.1 Simple socle submodules of $R[n]_I$

For the proof of theorem 5 in section 2.2 we need to look at submodules of $R[n]_I$ of length ≤ 3 with simple socle.

Proposition 9 *Let* $n \in \mathbb{N}_1$ *and* I *a subset of* $\mathbb{N}_1$. *Then* $R[n]_I$ *has precisely the following submodules* N *of length* 2 *or* 3 *with simple socle:*

(1) $|N| = 2$. *There are two types of such submodules:*

- (a) *$N = Aw$, where $w = \sum_{i=1, i\in I}^{n} \mu_i y_i$, $\mu_i \in k$ for $1 \leq i \leq n$, with $i \in I$ and not all $\mu_i = 0$.*
- (b) *$N = Aw$. If $1 \in I$, take $w = x_1 + \mu y_1$, $\mu \in k$. If $1 \notin I$, take $w = x_1$.*

(2) If $1 \in I$, then there exists a unique submodule N of length $|N| = 3$ with simple socle, namely $N = R[1]_I$. If $1 \notin I$, then there is no such submodule.

PROOF. For indecomposable submodules N of length 2 with $N \subset R[n]_I$ we are looking at a generator w of the top of N with a 1-dimensional image under α, β, γ, where $w = \sum_{j=1}^{n} \lambda_j x_j + \sum_{i=1, i\in I}^{n} \mu_i y_i$, with $\lambda_j, \mu_i \in k$, for $1 \leq i, j \leq n$, $i \in I$. Applying α, β, γ, we get $\alpha(w) = \sum_{j=1}^{n} \lambda_j z_j$, $\beta(w) = \sum_{j=2}^{n} \lambda_j z_{j-1}$, $\gamma(w) = \sum_{i=1, i\in I}^{n} \mu_i z_i$. Since we are looking for the submodules of length 2 we want this image $\langle \alpha(w), \beta(w), \gamma(w) \rangle$ to be 1-dimensional. This is the case when the associated matrix $R = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots\ldots & \lambda_n \\ \lambda_2 & \lambda_3 & \ldots\ \lambda_n & 0 \\ \mu_1 & \mu_2 & \ldots\ldots & \mu_n \end{pmatrix}$ has rank 1, where we define $\mu_i = 0$, for $1 \leq i \leq n$ and $i \notin I$. This can only happen in the following two cases, giving the possibilities for $N \subset R[n]_I$ with simple socle of length 2:

- (a) If for all $j = 1, \ldots, n$, $\lambda_j = 0$, and there exists at least one i, such that $\mu_i \neq 0$. Then $w = \sum_{i=1, i\in I}^{n} \mu_i y_i$, not all $\mu_i = 0$.
- (b) If $\lambda_j \neq 0$ for at least one j, then $j = 1$, since otherwise we can choose the maximal j, such that $\lambda_j \neq 0$. But for $j > 1$ we get $rk(R) > 1$, contradicting the fact that we need the rank to be 1 to get a 1-dimensioal image. Therefore $j = 1$.
  With $\lambda_1 \neq 0$ and if $1 \in I$, we can have μ_1 to be zero or non-zero, but $\mu_i = 0$ for all $i \in I \cap \{2, \ldots, n\}$: If not, i.e. $\mu_i \neq 0$ for at least one $i > 1$, $i \in I$, then we would also have $rk(R) > 1$, a contradiction. In this second case we therefore have $w = \lambda_1 x_1 + \mu_1 y_1$ with μ_1 possibly zero, and $\alpha(w) = \lambda_1 z_1$, $\beta(w) = 0$, $\gamma(w) = \mu_1 z_1$ or zero. W.l.o.g. we have $w = x_1 + \mu y_1$ for $\mu \in k$. In the case when $1 \notin I$, then $w = x_1$.

Note that every type (b) submodule is a submodule of $R[1]_I$. We need the following result:

Lemma 10

- *(i) Let N and N' be type (a) submodules of $R[n]_I$ with $N \cap N' \neq 0$. Then $N = N'$.*
- *(ii) Let N_1 be a type (a) and N_2 a type (b) submodule of $R[n]_I$ with $N_1 \cap N_2 \neq 0$. Then $1 \in I$ and $N_1 + N_2 = R[1]_I$.*
- *(iii) Let N and N' be type (b) submodules of $R[n]_I$ with $N \neq N'$ and $N \cap N' \neq 0$. Then $1 \in I$ and $N_1 + N_2 = R[1]_I$.*

PROOF. (i): Since N and N' are of type (a), they are of length 2, have simple socle, and can be written as $N = Aw$, where $w = \sum_{i=1, i\in I}^{n} \mu_i y_i,\ \mu_i \in k$ for $1 \leq i \leq n$, with $i \in I$ and not all $\mu_i = 0$. Similarly $N' = Aw'$, where $w' = \sum_{i=1, i\in I}^{n} \mu_i' y_i,\ \mu_i' \in k$ for $1 \leq i \leq n$, with $i \in I$ and not all $\mu_i' = 0$. We have $\gamma(w) = \sum_{i=1, i\in I}^{n} \mu_i z_i$, and not all $\mu_i = 0$. But $N \cap N' \neq 0$, so $\gamma(w)$ is a scalar multiple of $\gamma(w')$ and therefore $N = N'$.

(ii): Both N_1 and N_2 are of length 2 with simple socle. N_1 is of type (a), so $N_1 = Aw$, $w = \sum_{i=1, i\in I}^{n} \mu_i y_i,\ \mu_i \in k$ for $1 \leq i \leq n$, with $i \in I$ and not all $\mu_i = 0$. Since $N_1 \cap N_2 \neq 0$, and N_2 is of type (b), $N_2 = Aw'$, $w' = x_1 + \mu' y_1$, for $\mu' \in k$, we have $1 \in I$: If not, i.e. assume $1 \notin I$, so $w' = x_1$, then $\alpha(w') = \alpha(x_1) = z_1$, $\beta(w') = 0$, $\gamma(w') = 0$. This contradicts $N_1 \cap N_2 \neq 0$, since if $1 \notin I$, then $z_1 \notin N_1$. So $1 \in I$, $N_2 = A(x_1 + \mu' y_1)$ and since $|N_1| = 2$ and $N_1 \cap N_2 \neq 0$, we have $N_1 = Aw = A(\mu_1 y_1)$. Now $N_1 + N_2 = Aw + Aw' = A(x_1 + \mu' y_1) + A(\mu_1 y_1)$. So $N_1 + N_2$ has basis x_1, y_1, z_1, and $N_1 + N_2 = R[1]_I$.

(iii): Since N and N' are of type (b), they are of length 2 and have simple socle. Furthermore, since $N \neq N'$, they cannot both be equal to $A(x_1)$. So we must have $1 \in I$. W.l.o.g. we can write $N = Aw$, $w = x_1 + \mu y_1$ for $\mu \in k$ and $N' = Aw'$, $w' = x_1 + \mu' y_1$ for $\mu' \in k$ and $\mu' \neq \mu$, since $N \neq N'$. With $\alpha(w) = z_1$, we have that $N + N' = Aw + Aw'$ has basis x_1, y_1, z_1, thus $N_1 + N_2 = R[1]_I$.

To complete the proof of the proposition assume $1 \in I$. If N is a submodule of $R[n]_I$ with simple socle, such that $|N| = 3$, then $N = N_1 + N_2$, where $N_1, N_2 \subseteq N \subset R[n]_I$ are of length 2 with simple socle and $|N_1 \cap N_2| = 1$: Since N is a submodule of $R[n]_I$, let $N^{(1)} \subset R[n]_I^{(1)}$ and $N^{(2)} \subset R[n]_I^{(2)}$. N has length 3 and simple socle, so $N^{(1)}$ is one-dimensional and $N^{(2)}$ must have dimension 2, say $N^{(2)} = \langle x_1, x_2 \rangle$. Now consider Ax_1: x_1 is not in the socle, thus $Jx_1 \neq 0$, where J is the radical of A. Then $Jx_1 \subseteq N^{(1)}$, and since $N^{(1)}$ is one-dimensional, we have equality $Jx_1 = N^{(1)}$. So Ax_1 has length 2. Now, similarly, consider Ax_2 and conclude by letting $N_1 = Ax_1$ and $N_2 = Ax_2$, both of length 2 and $|Ax_1 \cap Ax_2| = 1$.

Lemma 10 (i) then implies that N_1 and N_2 cannot both be of type (a), since then we would have $|N| = 2$, a contradiction. Thus we must be in the case of either (ii) or (iii), implying $N_1 + N_2 = R[1]_I$ and so $N = R[1]_I$.

Lemma 11 *Let* $1 \leq t < n$ *and* $t + 1 \in I$. *Let* $V = R[t]_I + Aw$ *with* $w = x_{t+1} + \mu y_{t+1}$, *where* $\mu \in k$. *Then* $R[n]_I/V$ *has precisely one simple non-projective submodule, namely* $R[t+1]_I/V$.

PROOF. Since $t + 1 \in I$ and $V = R[t]_I + Aw$ with $w = x_{t+1} + \mu y_{t+1}$, $R[t+1]_I/V$ has length 1, i.e. $|R[t+1]_I/V| = 1$. To show the lemma, let us look at the possible submodules N of length 1 of $R[n]_I/V$ that are not simple projective. We are thus looking at a 1-dimensional top of N with generator u, such that we have $\alpha(u) = 0, \beta(u) = 0, \gamma(u) = 0$ in $R[n]_I/V$ and $|N| = 1$. We have $u = \sum_{j=t+2}^{n} \lambda_j x_j + \sum_{i=t+1, i\in I}^{n} \mu_i y_i$. The image of u under α, β, γ vanishes only in one single case: $\mu_{t+1} \neq 0$, and for $j \geq t+2$, $\lambda_j = 0$, and $\mu_i = 0$, for all $i \geq t+2$, $i \in I$. In all other cases the image would be at least one-dimensional. So $u = \mu_{t+1} y_{t+1}$, and $\alpha(u) = 0 = \beta(u)$ and $\gamma(u) = \mu_{t+1} z_{t+1}$ which equals 0 in the factor module $R[n]_I/V$. Hence $N = \langle y_{t+1} \rangle$, which is just $R[t+1]_I/V$.

2.2 Proof of Main Theorem

Assume $1 \in I$. Let us first collect some results:

Lemma 12 *Let $n \geq 2$. If V is a submodule of $R[n]_I$ which contains $U = R[n-1]_I$ and $|V| = |U| + 1$, then V is decomposable.*

PROOF. We need to show that there is no indecomposable submodule $V \subseteq R[n]_I$, such that $U \subseteq V$ and $|V/U| = 1$. Assume, to get a contradiction, V is indecomposable. Using the General Structure Lemma 8 we know that $R[n]_I/R[n-1]_I \cong R[1]_J$, where $J = \{i - n + 1 \,|\, i \in I, i > n-1\}$. So V/U is a one-dimensional submodule of $R[1]_J$ and using proposition 9 we get that it is generated by z_n, i.e. $V/U = \langle z_n \rangle$. But then the factor module V/U is simple projective giving a contradiction, since V was assumed indecomposable and not simple, so it cannot have a simple projective factor module. We conclude that such submodule V has to be decomposable.

Lemma 13 *Let $n \in I$, $n \geq 2$. Let $U = R[n-1]_I$ and V be a submodule of $R[n]_I$ of the form $V = U + Aw$ with $w = x_n + \mu y_n$, where $\mu \in k$. Then V is indecomposable.*

PROOF. Assume V is decomposable $V = V' \oplus V''$. Then the restriction of the reprsentation V to α and β (thus viewed as a representation over the Kronecker quiver), written $V|_{(\alpha,\beta)} = V'|_{(\alpha,\beta)} \oplus V''|_{(\alpha,\beta)}$, which is a decomposition of $V|_{(\alpha,\beta)}$. Then $V|_{(\alpha,\beta)} = X \oplus \langle y_j \,|\, 1 \leq j \leq n-1 \rangle$, where X is indecomposable. If $\mu = 0$, then $V = U + A(x_n)$ is $R[n]_J$, where $J = I \backslash \{n\}$. By proposition 4, $V = R[n]_J$ is indecomposable. In the case $\mu \neq 0$, we have $V|_{(\alpha,\beta)} \cong R[n]_J|_{\alpha,\beta}$. But $\langle y_j \,|\, 1 \leq j \leq n-1 \rangle$ is injective so then w.l.o.g. $V''|_{(\alpha,\beta)} \subseteq \langle y_j \,|\, 1 \leq j \leq n \rangle \subseteq R[n]_I^{(2)}$. So if V is decomposable then V would have a direct summand $V'' \subseteq \langle y_j \,|\, j \in I \rangle \subseteq V \subseteq R[n]_I$, contradicting that V, as a submodule of $R[n]_I$, has no direct summand isomorphic to the injective module S_2 over $K(3)$. We conclude $V = U + Aw$ with $w = x_n + \mu y_n$, $\mu \in k$, is indecomposable.

Lemma 14 *Let $n \in I$, $n \geq 2$. The maximal indecomposable submodules of $R[n]_I$ which contain $U = R[n-1]_I$, are the submodules of the form $U + Aw$ with $w = x_n + \mu y_n$, where $\mu \in k$.*

PROOF. Let V be a maximal indecomposable submodule of $R[n]_I$ which contains $U = R[n-1]_I$. We will show that $V = U + Aw$ with $w = x_n + \mu y_n$, where $\mu \in k$.

Let us determine the possibilities for the factor module V/U having length 2, using proposition 9. First note that V/U is of type (b). Assume, to get a contradiction, V/U is of type (a). Then since $U \subset V$, $V = U + Aw$, with $w \in \langle x_j, y_i | j, i \geq n,\, i \in I\rangle$, in the following equation $w = \sum_{i \geq n, i \in I} \mu_i y_i$, ($\mu_i \in k$) there exists at least one i, such that $\mu_i \neq 0$. Then $\alpha(w) = 0 = \beta(w)$, but $\gamma(w) = \sum_{i \geq n, i \in I} \mu_i z_i$, not all $\mu_i = 0$. With $U = R[n-1]_I$ having basis x_j, y_i, z_j, $1 \leq j, i \leq n-1$, $i \in I$, we would get a direct sum $V = U \oplus Aw$ giving decomposable V, a contradiction. So V/U is of type (b), and since $n \in I$, $V = U + Aw$ with $w = x_n + \mu y_n$, $\mu \in k$.

Lemma 15 *Let $1 \leq t < n$ and $U = R[t]_I$ be a submodule of $R[n]_I$. If V is an indecomposable submodule of $R[n]_I$ which contains U such that $|V/U| = 2$, then V is a submodule of $R[t+1]_I$ and has the form $V = U + Aw$ with $w = x_{t+1} + \mu y_{t+1}$, $\mu \in k$, if $t+1 \in I$. If $t+1 \notin I$, then $V = U + Aw$ with $w = x_{t+1}$, so $V = R[t+1]_I$.*

PROOF. By the General Structure Lemma 8 we know that $R[n]_I / R[t]_I \cong R[n-t]_J$, where $J = \{i - t \,|\, i \in I, i > t\}$. So V/U is a length 2 submodule of $R[n-t]_J$, so we can apply proposition 9: Note that V/U is of type (b). If not, i.e. V/U is of type (a), then since $U \subset V$, $V = U + Aw$, with $w \in \langle x_j, y_i | t+1 \leq j, i \leq n,\, i \in I\rangle$, in the following equation $w = \sum_{i \geq t+1, i \in I}^{n} \mu_i y_i$, ($\mu_i \in k$) there exists at least one i, such that $\mu_i \neq 0$. Then $\alpha(w) = 0 = \beta(w)$, but $\gamma(w) = \sum_{i = t+1, i \in I}^{n} \mu_i z_i$, not all $\mu_i = 0$. With $U = R[t]_I$ having basis x_j, y_i, z_j, $1 \leq j, i \leq t$, $i \in I$, we would get a direct sum $V = U \oplus Aw$ giving decomposable V, a contradiction. So V/U is of type (b), i.e. if $t+1 \in I$, then $V = U + Aw$ with $w = x_{t+1} + \mu y_{t+1}$, $\mu \in k$, an in the case $t+1 \notin I$, then $V = U + Aw$ with $w = x_{t+1}$, so $V = R[t+1]_I$

Corollary 16 *Let $t < n$. If $R[t]_I$ is the unique piling submodule of $R[n]_I$ of length $|R[t]_I|$ and $t+1 \in I$, then the maximal submodules of $R[t+1]_I$ that contain $U = R[t]_I$ and are of the form $U + Aw$ with $w = x_{t+1} + \mu y_{t+1}$, where $\mu \in k$, are piling submodules of $R[n]_I$.*

PROOF. Let V be of the form $U + Aw$ with $w = x_{t+1} + \mu y_{t+1}$, where $\mu \in k$, and a maximal submodule of $R[t+1]_I$ that contains $U = R[t]_I$. From lemma 13 we know that V is indecomposable and $|V| = |U| + 2$. Since $U = R[t]_I$ is a piling submodule of $R[n]_I$ and V is indecomposable, we know by lemma 12 that there is no indecomposable submodule $U' \subset V$ with $U \subset U'$ and $|U'| = |U| + 1$. Therefore by proposition 7, V is a piling submodule of $R[n]_I$.

Corollary 17 *Let* $1 \leq t < n$. *Suppose that* $U = R[t]_I$ *is the unique piling submodule of* $R[n]_I$ *of length* $|R[t]_I|$. *Then any maximal indecomposable submodule* V *containing* U, *such that* $|V| = |U| + 2$, *is piling and of the following form:*

(1) If $t + 1 \in I$, *then* $V = U + Aw$ *with* $w = x_{t+1} + \mu y_{t+1},\ \mu \in k$.
(2) If $t + 1 \notin I$, *then* $V = U + Aw$ *with* $w = x_{t+1}$, *i.e.* $V = R[t + 1]_I$.

In particular, if $t + 1 \notin I$, *then* $R[t + 1]_I$ *is the only piling submodule of length* $|R[t + 1]_I|$ *that contains* $R[t]_I$.

PROOF. This follows from the above lemmas. By lemma 15, since V is an indecomposable submodule of $R[n]_I$ containing U with $|V/U| = 2$, we have that V is a submodule of $R[t + 1]_I$ of the desired form: if $t + 1 \in I$, then $V = U + Aw$ with $w = x_{t+1} + \mu y_{t+1},\ \mu \in k$, and if $t + 1 \notin I$, then $V = U + Aw$ with $w = x_{t+1}$. By corollary 16, V is piling. In the second case, if $t+1 \notin I$, then using 14, $V = U+Aw$ with $w = x_{t+1}$, i.e. $V = R[t+1]_I$, which is just the type (b) submodule of proposition 9. Since this is indecomposable and has length $|U| + 2$, it is piling by proposition 7 (as there is no indecomposable submodule of length $|U| + 1$ by 12).

PROOF. [Proof of the Main Theorem 5] The proof is done by induction: assume U is a piling submodule of $R[n]_I$ of length at least 3. If $|U| = |R[m]_I|$ for some $1 \leq m \leq n$, then we will show by induction on m that $U = R[m]_I$. This is part (1) of the theorem.

For $m = 1$, since $1 \in I$, we know by proposition 9 that $R[1]_I$ is the only length 3 submodule of $R[n]_I$ with simple socle, hence a piling submodule. If U is a piling submodule of $R[n]_I$ of length $|U| = 3 = |R[1]_I|$, it must have simple socle by proposition 9, thus $U = R[1]_I$. This completes the base case. Now we let $1 \leq t < n$ and assume that $U = R[t]_I$ is the unique piling submodule of $R[n]_I$ of length $|R[t]_I|$. Let us show that $R[t + 1]_I$ is the unique piling submodule of $R[n]_I$ of length $|R[t + 1]_I|$. This is done in two steps: (a) $R[t + 1]_I$ is piling, and (b) if W is piling of length $|R[t + 1]_I|$, then $W = R[t + 1]_I$.

(a) If $t + 1 \in I$, using lemma 14, we know that, the maximal indecomposable submodules of $R[t + 1]_I$, that contain $U = R[t]_I$, are of the form $U + Aw$, with $w = x_{t+1} + \mu y_{t+1}$ ($U + Ay_{t+1}$ cannot occurr, since then it would be decomposable, as in lemma 14). By lemma 16, the maximal submodules of

$R[t+1]_I$, that contain $U = R[t]_I$ are piling submodules. By lemma 12, there are no piling submodules V of length $|R[t]_I| + 1$, but there are indecomposable submodules V of length $|R[t]_I| + 2$ containing $R[t]_I$. Using proposition 7, lemma 15 and corollary 16, those are the piling submodules of length $|R[t]_I| + 2$ and all of them are submodules of $R[t+1]_I$. Since $t+1 \in I$, every such V has the form $U + Aw,$ with $w = x_{t+1} + \mu y_{t+1}, \mu \in k$. Now, $R[t+1]_I$ is indecomposable (by proposition 4) and $R[t+1]_I$ contains V with $|R[t+1]_I| = |V| + 1$. Thus by proposition 7, since V is piling, $R[t+1]_I$ is piling too. If $t+1 \notin I$, then there is only one piling submodule of $R[t+1]_I$ of length $|R[t+1]_I|$, nameley $R[t+1]_I$ itself, since in this case $R[t+1]_I = R[t]_I + Ax_{x_{t+1}}$, which is piling by corollary 17.

To show part (b), note that by corollary 17, we know that V, such that $U \subset V \subseteq R[n]_I$ and $|V/U| = 2$, is piling and has the form $V = R[t]_I + Aw$, with $w = x_{t+1} + \mu y_{t+1},\ \mu \in k$, provided $t+1 \in I$. In the case $t+1 \notin I$, $w = x_{t+1}$. Furthermore we have that $V \subseteq R[t+1]_I$ and is is indecomposable.
(b) Let W be a piling submodule of $R[n]_I$ of length $|R[t+1]_I|$. Since both W and $R[t+1]_I$ are piling submodules of the same length, W has piling submodules U' of length $|R[t]_I|$ and V' of length $|R[t]_I| + 2$, with $U' \subset V' \subset W$. By induction hypothesis, $R[t]_I$ is the unique piling submodule of length $|R[t]_I|$, so we have $U' = R[t]_I = U$. Also, by lemma 15, V' is a submodule of $R[t+1]_I$ of the form $U + Aw,$ with $w = x_{t+1} + \mu y_{t+1}, \mu \in k$, provided $t+1 \in I$. So we are left to show, that if $t+1 \in I,$ $R[t+1]_I$ is the only indecomposable submodule of $R[n]_I$, having length $|R[t+1]_I|$ and contains $V' = U + Aw$, with $w = x_{t+1} + \mu y_{t+1}$. By lemma 11, we have that both $R[n]_I/V$ and $R[n]_I/V'$ have precisely one simple non-projective submodule, which is $R[t+1]_I/V$. Since V' and V have length $|R[t]_I| + 2$, W has length $|W| = |V'| + 1 = |R[t]_I| + 3$. Now, $W = R[t+1]_I$ follows from the General Structure Lemma 8: we know that $R[n]_I/R[t]_I \cong R[n-t]_J$, where $J = \{i - t \,|\, i \in I, i > t\}$. We will again use proposition 9. Since $1 \in I$ and W/U has simple socle and is of length $|W/U| = 3$, part 3 of propostion 9 states that if $1 \in J$, the only submodule of length 3 of $R[n-t]_J$ is $R[1]_J$. For $t+1 \in I$, $W = V' + Ay_{t+1}$ and $R[t]_I \subset W \subset R[n]_I$ with $|W| = |R[t]_I| + 3 = |R[t+1]_I|$, we have $1 \in J$ and conclude $W = R[t+1]_I$, for $t+1 \in I$. In the case $t+1 \notin I$, we have already seen that $R[t+1]_I = R[t]_I + Ax_{x_{t+1}} = V'$ is the only piling submodule of of length $|R[t+1]_I|$, which in this case is just V.
So $R[t+1]_I$ is the unique piling submodule of $R[n]_I$ of length $|R[t+1]_I|$. This finishes part (1) of theorem 5.

Let us prove part (2) and (3) of the theorem. Let L be a piling submodule of $R[n]_I$, of length $|L| \neq |R[m]_I|$, for all $m \leq n$. Choose m maximal, such that $|R[m]_I| \leq |L|$. Then, by part (1) of the theorem, we have that L has a piling submodule of length $|R[m]_I|$, thus $R[m]_I$ is a piling submodule of L. By lemma 12, the length of L cannot be $|R[m]_I| + 1$, since else L would be decomposable. We must have $|L| = |R[m]_I| + 2$, which follows from the choice of m and the following considerations: if $|L| = |R[m]_I| + 3 = |R[m+1]_I|$ and $m \in I$, then by part (1) of the theorem, $L = R[m+1]_I$, contradicting the fact that L is a piling submodule of length $|L| \neq |R[m]_I|$, for all $m \leq n$. If $m \notin I$, then if $|L| = |R[m]_I| + 2 = |R|m+1]_I|$, again $L = R[m+1]_I$. So we have $m \in I$ and $|L| = |R[m]_I| + 2$. By lemma 15, L is a submodule of $R[m+1]_I$ and has the form $R[m]_I + A(x_{m+1} + \mu y_{m+1})$, $\mu \in k$, since $m+1 \in I$. Such submodules are maximal by lemma 14 and piling submodules by corollary 16 and corollary 17. Since $\mu \in k$, we have a one-parameter family of piling submodules of length $|R[m+1]_I| - 1$. This completes part (2), and since L was chosen to be a piling submodule of $R[n]_I$, of length $|L| \neq |R[m]_I|$, for all $m \leq n$, we also have shown that there are no other piling submodules of length at least 3 than those stated in (1) and (2) of the theorem. This finishes the proof of theorem 5.

Note that all submodules of $R[n]_I$ occurring in a Gabriel-Roiter filtration of $R[n]_I$ are piling submodules. We can now prove corollary 6, stating that any piling submodule of $R[n]_I$ of length at least 3 belongs to a Gabriel-Roiter filtration of $R[n]_I$.

PROOF. [Proof of corollary 6] Let $M_1 \subset M_2 \subset \ldots \subset M_r$ be a Gabriel-Roiter filtration of $R[n]_I$ with Gabriel-Roiter measure $\mu(R[n]_I) = \{l_1, l_2, \ldots, l_r\}$, where $l_i = |M_i|$ for $1 \leq i \leq r$, with $l_1 < l_2 < \ldots < l_r$ and $l_r = |R[n]_I|$. Each M_i is a piling submodule of $R[n]_I$. If $l_i = |R[m]_I|$ for some $1 \leq m \leq n$, then theorem 5 part (1) implies $V = R[m]_I$, so $M_i = R[m]_I = V$. Let V be a piling submodule of $R[n]_I$, such that $V \neq R[m]_I$, for $1 \leq m \leq n$. Then theorem 5 part (2) implies that there exists an $1 \leq m \leq n$ with $m+1 \in I$, such that $V = R[m]_I + A(x_{m+1} + \mu y_{m+1})$, $\mu \in k$, and V has a piling submodule so that $R[m]_I \subset V \subset R[m+1]_I$. Then there exists i, j, with $i < j < n$, such that $M_i = R[m]_I$ and $M_j = R[m+1]_I$. By lemma 12, there is no indecomposable submodule of $R[m+1]_I$ of length $|R[m]_I| + 1$. Since $|V| = |R[m]_I| + 2 = |R[m+1]_I| - 1$, as $m+1 \in I$, and $|V| = |M_j| - 1$ we have that $j = i+2$ and $|M_{i+1}| = |V|$. Hence the chain $M_1 \subset M_2 \subset \ldots \subset M_i \subset V \subset M_{i+2} \subset \ldots \subset R[n]_I$ is a Gabriel-Roiter filtration of $R[n]_I$.

2.3 The Gabriel-Roiter measure depends on I

Proposition 18 *If* $\mu(R[n]_I) = \mu(R[n]_J)$, *then* $I \cap \{1, \ldots, n\} = J \cap \{1, \ldots, n\}$.

PROOF. Recall that $R[n]_I$ depends on $I \cap \{1, 2, \ldots, n\}$. If $R[n]_I$ and $R[n]_J$ have the same Gabriel-Roiter measure, we will show by induction on t that $I \cap \{1, \ldots, t\} = J \cap \{1, \ldots, t\}$ for $1 \leq t \leq n$. The base case is clear, since if $t = 1$, then $I \cap \{1\} = \{1\} = J \cap \{1\}$, since $1 \in I$ and $1 \in J$. So we are in the case of the unique piling submodule of length 3 with simple socle, $R[1]_I$, as in section 2.1, whose Gabriel-Roiter measure is $\{1, 2, 3\}$. Assume that for some t we have $I \cap \{1, \ldots, t\} = J \cap \{1, \ldots, t\}$. Now look at the case $t + 1 \leq n$, $I \cap \{1, \ldots, t + 1\}$. Let $t + 1 \in I$. Then we know from theorem 5 that $R[n]_I$ has piling submodules M, of length $|R[t + 1]_I|$, and N of length $|R[t + 1]_I| - 1$. Thus $\mu(M) = \mu(R[n]_I) \cap \{1, \ldots, |R[t + 1]_I|\}$ and $\mu(N) = \mu(R[n]_I) \cap \{1, \ldots, |R[t + 1]_I| - 1\}$. Since $\mu(R[n]_I) = \mu(R[n]_J)$, $R[n]_J$ must have the same piling submodule structure, by definition of the Gabriel-Roiter measure and piling submodules. So $R[n]_J$ has piling submodules of length $|R[t+1]_J|$ and $|R[t+1]_J|-1$. If $t+1 \notin J$, then $J\cap\{1, \ldots, t+1\} = J\cap\{1, \ldots, t\} = I \cap \{1, \ldots, t\}$, by induction, but then using theorem 5, $R[n]_J$ has no piling submodules of length $|R[t+1]_J|-1$, giving a contradiction. Therefore $t+1 \in J$.

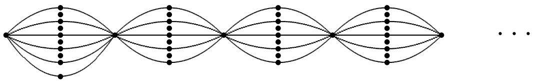
2.4 Case $I = \mathbb{N}_1$

Let us quickly look at the case when $I = \mathbb{N}_1$. Let $n \in \mathbb{N}_1$ and denote the corresponding representation by $R[n]_{\mathbb{N}_1}$. Let $n \in \mathbb{N}_1$. Using theorem 5 one can show that for any $1 \leq m \leq n$, $R[n]_{\mathbb{N}_1}$ has a unique piling submodule of length $|R[m]_{\mathbb{N}_1}| = 3m$, which is the submodule $R[m]_{\mathbb{N}_1}$. Also for any $2 \leq m \leq n$, there exists a one-parameter family of piling submodules of length $|R[m]_{\mathbb{N}_1}|-1 = 3m-1$, namely the maximal submodules of $R[m]_{\mathbb{N}_1}$ that contain $R[m - 1]_{\mathbb{N}_1}$. One can give a closed formula for the Gabriel-Roiter measure of $R[n]_{\mathbb{N}_1}$ using the following considerations. The Gabriel-Roiter measure of $R[m + 1]_{\mathbb{N}_1}$ is $\mu(R[m + 1]_{\mathbb{N}_1}) = \mu\{R[m]_{\mathbb{N}_1}\} \cup \{3m + 2, 3m + 3\}$. With an induction as in theorem 5 we get $\mu(R[n]_{\mathbb{N}_1}) = \{1, 3l - 1, 3l \,|\, 1 \leq l \leq n\}$, since $|R[m+1]_{\mathbb{N}_1}| = 3m+3 = 3(m+1)$ and $|R[m]_{\mathbb{N}_1}|+2 = 3m+2 = |R[m+1]_{\mathbb{N}_1}|-1$.

Let us give a name to a class of unique piling submodules.

Definition 19 *An indecomposable submodule U of a module M is called a* knotted module*, provided U is the unique piling submodule of M of length $|U|$.*

For $1 \leq m \leq n$, $R[n]_{\mathbb{N}_1}$ is a knotted module, since it has a unique piling submodule of length $|R[m]_{\mathbb{N}_1}| = 3m$, which is the submodule $R[m]_{\mathbb{N}_1}$. Since we have seen that for any $1 \leq m \leq n$, there exists a one-parameter family of piling submodules of length $|R[m+1]_{\mathbb{N}_1}| - 1 = 3m - 1$, one can picture $R[n]_{\mathbb{N}_1}$ as follows: The one-parameter family of maximal submodules of $R[m]_{\mathbb{N}_1}$ that contain $R[m]_{\mathbb{N}_1}$ are drawn as dots between the knotted modules. Note that in the case of $R[1]_{\mathbb{N}_1}$, one has the additional Gabriel-Roiter submodule generated by $\langle y_1 \rangle$ of length 2. Here is an example how one could picture the knotted module $R[n]_{\mathbb{N}_1}$:



2.5 Gabriel-Roiter formulae for $R[n]_I$

Let $I \subset \mathbb{N}_1$ and $1 \in I$ and let us come back to the more general situation with representations $R[n]_I$. We have $\min\{\mathbf{dim}\, R[n]_I\} = (n, n+1)$ and $\max\{\mathbf{dim}\, R[n]_I\} = (n, 2n)$. Let $n < d \leq 2n$, then $\min\{q((n,d))\} = q((n,2n)) = q((n,n)) = -n^2$, $\max\{q((n,d))\} = q((n, \lfloor 3n/2 \rfloor)) = \lfloor -5n^2/4 \rfloor$, and the length of $R[n]_I$ is a positive integer between $2n+1$ and $3n$.

The Gabriel-Roiter measure of $R[n]_I$ depends on $I \cap \{1, 2, \ldots, n\}$ and we have seen in the previous sections the case $I = \mathbb{N}_1$. Let us now look at the case when $I = \{1\}$. Then $R[n]_I$ has dimension vector $\mathbf{dim}\, R[n]_I = (n, n+1)$. The Gabriel-Roiter measure of $R[n]_I$ is $\mu(R[n]_{\{1\}}) = \{1, 2, \ldots, 2i-1, \ldots, 2n+1\}$ for $1 < i \leq n$, $2n+1$ being the length of $R[n]_I$. Finally, applying theorem 5 together with its corollary we can give a recursive formula for the Gabriel-Roiter measure of $R[n]_I$ in general.

Theorem 20 *Let $I \subset \mathbb{N}_1$ and $1 \in I$. Let $n \in \mathbb{N}_1$ and $R[n]_I$ a representation of length $r \geq 3$ with Gabriel-Roiter filtration $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_r = R[n]_I$. Let $\{l_1, l_2, l_3, \ldots, l_r\}$ be the Gabriel-Roiter measure of $R[n]_I$, where $l_i = |W_i|$, for $1 \leq i \leq r$. Then*

(i) $l_1 = 1, l_2 = 2, l_3 = 3$, *i.e. the Gabriel-Roiter measure of $R[n]_I$ starts with* $\{1, 2, 3\}$.

(ii) *for* $i \geq 3$, $l_i = l_{i-1} + \begin{cases} 2, \text{ if } l_{i-1} - l_{i-2} = 1 \text{ or } \dim(\operatorname{soc} W_{i-1}) + 1 \notin I \\ 1, \text{ else} \end{cases}$

2.6 Uncountably many Gabriel-Roiter measures

Recall theorem 2, stating that any module M with a Gabriel-Roiter filtration is indecomposable. We will use this result to extend our results from previous sections on finitely generated modules to infinitely generated modules. We now consider an infinitely generated module $R[\infty]_I$ for a given set $I \subseteq \mathbb{N}_1$. Define the representation $R[\infty]_I$ for the quiver $K(3)$ analogous to the finitely generated case: the vector space $R[\infty]_I^{(1)}$ has basis z_i, $i = 1, 2, \ldots$, the vector space $R[\infty]_I^{(2)}$ has basis x_j, y_i, where $j = 1, 2, \ldots$, and $i \in I$, which can be an finite or infinite set. Let $\alpha(x_j) = z_j, \beta(x_j) = z_{j-1}$, and $\gamma(x_j) = 0$,

$\alpha(y_i) = 0, \beta(y_i) = 0,$ and $\gamma(y_i) = z_i$, for all $i, j,$ with $i \in I$, and where $z_0 = 0$. Define $R[0]_I = 0$. Again we assume $1 \in I$.

Remark 1 *The module M_i, for $i \geq 2$, of the infinite Gabriel-Roiter filtration $M_1 \subset M_2 \subset M_3 \subset \ldots \subset \bigcup_n M_n = R[\infty]_I$ lies in the central part of* $\operatorname{mod} A$ *and $R[\infty]_I$ is an infinite-dimensional module in the central part.*

Let $I \subseteq \mathbb{N}_1$ with $1 \in I$. The classification of piling submodules in theorem 5 also holds for the infinitely generated module $R[\infty]_I$: Every piling submodule is finitely generated and a submodule of $R[n]_I$ for some $n \in \mathbb{N}_1$. We have inclusion maps $R[0]_I \subset R[1]_I \subset R[2]_I \subset \ldots \subset R[n]_I \subset \ldots \subset R[\infty]_I$. Our theorem 5 extends to this case:

Theorem 21 *Let $I \subseteq \mathbb{N}_1$ with $1 \in I$. Then the piling submodules of $R[\infty]_I$ of length at least 3 are of two kinds:*

(1) *For any $m \in \mathbb{N}_1$ there exists a unique piling submodule of length $|R[m]_I|$, which is the submodule $R[m]_I$.*
(2) *For any $2 \leq m \in \mathbb{N}_1$ with $m \in I$, there exists a one-parameter family of piling submodules of length $|R[m]_I| - 1$, namely the maximal submodules of $R[m]_I$ that contain $R[m-1]_I$.*

Theorem 22 *There are uncountably many Gabriel-Roiter measures for 3-Kronecker quiver.*

PROOF. Let $I \subseteq \mathbb{N}_1$ containing 1. There is a one-to-one correspondence between I and $R[\infty]_I$, since different I induce non-isomorphic $R[\infty]_I$. Furthermore, different subsets $I \neq J$ of $\mathbb{N}_1$ yield different Gabriel-Roiter measures, by proposition 18, so $R[\infty]_I$ and $R[\infty]_J$ have different Gabriel-Roiter measures. Thus we get uncountably many Gabriel-Roiter measures $\mu(R[\infty]_I)$, for $I \subseteq \mathbb{N}_1$.

Corollary 23 *The central part of* $\operatorname{mod} A$ *contains uncoutably many Gabriel-Roiter measures. Thus there exists uncountably many indecomposable infinite-dimensional modules in the central part of* $\operatorname{mod} A$.

3 Possible largest central Gabriel-Roiter measure

The following is a theorem of C.M. Ringel [Ri06] for any algebra A of infinite representation type.

Theorem 24 *Let A be a finite-dimensional algebra of infinite representation type. Then there are Gabriel-Roiter measures I_t, I^t for A (with $t \in \mathbb{N}_1$) such* *that $I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$ and such that any other Gabriel-Roiter measure I for A satisfies $I_t < I < I^t$ for all $t \in \mathbb{N}_1$. Moreover, all these Gabriel-Roiter measures I_t and I^t are of finite type, i.e. there are only finitely many indecomposable modules haven these Gabriel-Roiter measures.*

The indecomposable modules corresponding to the measures I_t lie in the so-called *take-off part* of the module category $\operatorname{mod} A$, and those corresponding to the measures I^t are said to form the *landing part* of $\operatorname{mod} A$.

3.1 Slope and dimension vectors

The Tits form for the 3-Kronecker quiver for a dimension vector (x, y) is given by $q((x, y)) = x^2 + y^2 - 3xy$. Solving the equation $q((x, y)) = 0$ one gets the two possible solutions for x : $x_1 = (\frac{3+\sqrt{5}}{2})y$, and $x_2 = (\frac{3-\sqrt{5}}{2})y$. Thus we get two lines in the plane, one with slope $\frac{3+\sqrt{5}}{2}$, the other with slope $\frac{3-\sqrt{5}}{2}$. These lines are the asymptotes for the equation $q((x, y)) = 1$. The integral vectors d with $q(d) \leq 0$ are called *imaginary roots*, those with $q(d) = 1$ are called *real roots*. For any positive real root d, there is an indecomposable module M with $\mathbf{dim}\, M = d$, and this module is unique up to isomorphism according to Kac. We call those modules *real root modules*. The real roots for the 3-Kronecker quiver are:

$$\begin{gathered}(1,0),\ (3,1),\ (8,3),\ (21,8),\ (55,21),\ \ldots \\ (0,1),\ (1,3),\ (3,8),\ (8,21),\ (21,55),\ \ldots\end{gathered}$$

The upper sequence gives the dimension vectors of the indecomposable preprojective modules. The lower sequence gives the dimension vectors of the indecomposable preinjective modules. The imaginary roots of the 3-Kronecker quiver are all $(n, m) \in \mathbb{N}_1^2$ with $\frac{3-\sqrt{5}}{2} < \frac{n}{m} < \frac{3+\sqrt{5}}{2}$. Drawing these dimension vectors in the plane one sees that the preinjective dimension vectors just lie below the line with slope $\frac{3-\sqrt{5}}{2} = 1 + \sqrt{5} - \varphi$. We are interested in those dimension vectors which lie just above this line. These lie on the infinite sequence $\mathcal{D}$ of dimension vectors:

$$\mathcal{D}: \quad \binom{1}{1}, \binom{2}{1}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \binom{6}{3}, \binom{7}{3}, \binom{8}{4}, \binom{9}{4}, \binom{10}{4}, \binom{11}{5}, \binom{12}{5}, \binom{13}{5}, \binom{14}{6}, \binom{15}{6}, \ldots$$

Let us denote by $\mathbf{d}_n$ the nth dimension vector in the above list, i.e. the dimension vector at position n, which is just the upper entry of $\mathbf{d}_n = \binom{n}{m} \in \mathbb{N}^2$. These dimension vectors occur naturally as imaginary roots of the 3-Kronecker quiver and the sequence $\mathcal{D}$ can be described as follows: For every n, m is the smallest value, such that $\frac{n}{m} < \frac{3+\sqrt{5}}{2}$. That means, for any other dimension vector $\binom{n}{m'} \in \mathbb{N}^2$, such that $\frac{n}{m'} < \frac{3+\sqrt{5}}{2}$, we have $m \leq m'$.

One hopes to be able to construct indecomposable representations M_n for each of the dimension vectors $\mathbf{d}_n$ in the sequence $\mathcal{D}$. We then have for a module M with $\mathbf{dim}(M) = \mathbf{d_n}$, $n = \dim \operatorname{top}(M)$. Describing properties and a formula for the sequence $\mathcal{D}$ will be done in this section, construction of all the corresponding indecomposable modules is still an open problem. Let us compute the dimension vector $\mathbf{d}_n$ in $\mathcal{D}$ for every n.

Lemma 25 *For every* $n \in \mathbb{N}$, $\mathbf{d}_n = \binom{n}{2n-\lfloor n\varphi \rfloor}$, *where* $\varphi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio and* $\lfloor\ \rfloor$ *denotes the floor function.*

PROOF. Thus for a dimension vector $\binom{n}{m}$ we need to show $m = 2n - \lfloor n\varphi \rfloor$. Let $\binom{n}{y}$ be the point lying on the line $q((x,y)) = 0$. Then $y = n\frac{3-\sqrt{5}}{2}$, since $\frac{3-\sqrt{5}}{2}$ is the slope of the line. But the positive integer m is just the next integer above y, thus, when using the ceiling function, can be written as $m = \lceil n\frac{3-\sqrt{5}}{2} \rceil$. With $\frac{3-\sqrt{5}}{2} = 1 + \varphi - \sqrt{5}$ we get $m = \lceil n(1+\varphi-\sqrt{5}) \rceil = n + \lceil n(\varphi - \sqrt{5}) \rceil = n + \lceil n/2 - n\sqrt{5}/2 \rceil = 2n + \lceil -n/2 - n\sqrt{5}/2 \rceil$, with the last equality because the difference between $\lceil -n/2 - n\sqrt{5}/2 \rceil$ and $\lceil n/2 - n\sqrt{5}/2 \rceil$ is just n. Now $m = 2n + \lceil -n/2 - n\sqrt{5}/2 \rceil = 2n + \lceil -n(1+\sqrt{5})/2 \rceil = 2n + \lceil -n\varphi \rceil = 2n - \lfloor n\varphi \rfloor$, since for any $x \in \mathbb{R}$, $\lceil x \rceil = -\lfloor -x \rfloor$.

Calculating the Tits form for the dimension vectors in $\mathcal{D}$ one obtains:

$\mathbf{d}_n$	$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{2}$	$\binom{4}{2}$	$\binom{5}{2}$	$\binom{6}{3}$	$\binom{7}{3}$	$\binom{8}{4}$	$\binom{9}{4}$	$\binom{10}{4}$	$\binom{11}{5}$	$\binom{12}{5}$	$\binom{13}{5}$ …
$q(\mathbf{d}_n)$	-1	-1	-5	-4	-1	-9	-5	-16	-11	-4	-19	-11	-1 …

Remark 2 *The sequence of values of* $q(\mathbf{d}_n)$, $-1, -1, -5, -4, -1, -9, -5, \ldots$, *is a self-repeating sequence having many nice properties. It is sequence number* A005752 *in N. J. A. Sloane's* The On-Line Encyclopedia of Integer Sequences[1]. *This sequence is also closely related to the* Lower & Upper Wythoff sequence, *which is sequence number* A000201 *resp.* A001950. *It further turns out that the Wythoff sequences partition the dimension vectors in* $\mathcal{D}$ *into dimension vectors of knotted modules and non-knotted modules. These sequences are also Beatty sequences, i.e. they partition the natural numbers. Further investigation is needed to fully understand the representation-theoretical interpretation of these links.*

Proposition 26 *The value of the Tits form of a dimension vector* $\mathbf{d}_n \in \mathcal{D}$ *is* $q(\mathbf{d}_n) = -n^2 - n\lfloor n\varphi \rfloor + \lfloor n\varphi \rfloor^2$, *where* $\varphi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio and* $\lfloor\ \rfloor$ *denotes the floor function.*

PROOF. We have $q(\mathbf{d}_n) = q(\binom{n}{m}) = n^2 + m^2 - 3nm$ and from lemma 25 $m = 2n - \lfloor n\varphi \rfloor$. So $q(\mathbf{d}_n) = n^2 + (2n - \lfloor n\varphi \rfloor)^2 - 3n(2n - \lfloor n\varphi \rfloor) = -n^2 - n\lfloor n\varphi \rfloor + \lfloor n\varphi \rfloor^2$.

---

[1] `www.research.att.com/~njas/sequences`

Remark 3 *Of course one can generalise the above formulae for* n*-Kronecker quivers, taking into account the slope of the line defined by* $q(\mathbf{d}) = 0$*. For example, for the* 4*-Kronecker quiver the* n*th dimension vector in the analogous defined sequence* $\mathcal{D}_4$ *is* $\binom{n}{2n-\lfloor n\sqrt{3}\rfloor}$*. However, the* 3*-Kronecker quiver is special in a way, since this is the only case where the Golden Ratio plays a role.*

We conjecture the existence of an infinite-dimensional module M_∞ with infinite Gabriel-Roiter filtration $M_0 \subset M_1 \subset \ldots \subset M_n \subset \ldots$ and $\mathbf{dim}(M_n) = \mathbf{d}_n \in \mathcal{D}$, for all $n = 1, 2, \ldots$. If such a module M_∞ exists, we give a formula for the Gabriel-Roiter measure of an indecomposable module M of dimension vector $\mathbf{d}$ lying in the sequence $\mathcal{D}$.

Theorem 27 *Assume there is an infinite-dimensional module* M_∞ *with infinite Gabriel-Roiter filtration* $M_0 \subset M_1 \subset \ldots \subset M_n \subset \ldots$ *and* $\boldsymbol{dim}(M_n) = \mathbf{d}_n \in \mathcal{D}$*, for all* $n = 1, 2, \ldots,$ *where* $\mathcal{D}$ *is the infinite sequence of dimension vectors*

$$\binom{1}{1}, \binom{2}{1}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots, \binom{n}{2n-\lfloor n\varphi\rfloor}, \ldots.$$

Then the Gabriel-Roiter measure of an indecomposable module M_n *of dimension vector* $\mathbf{d}_n$ *is* $\{1, l_1, l_2, l_3, \ldots, l_n\}$*, where* $l_i = 3i - \lfloor i\varphi\rfloor$*, for* $1 \le i \le n$*, and* $\varphi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio and* $\lfloor\ \rfloor$ *denotes the floor function.*

PROOF. Recall from corollary 25 the ith dimension vector in the sequence $\mathcal{D}$, which is $\binom{i}{2i-\lfloor i\varphi\rfloor}$. The length of the module M_i with $\mathbf{dim}(M_i) = \mathbf{d}_i = \binom{n}{m}$ is $l_i = |M_i| = n + m = i + 2i - \lfloor i\varphi\rfloor = 3i - \lfloor i\varphi\rfloor$.

We want to conjecture the existence of a largest Gabriel-Roiter measure in the central part and that the infinite-dimensional module $M_\infty = \bigcup M_n$, with $\mathbf{dim}(M_n) = \mathbf{d}_n \in \mathcal{D}$, for all $n = 1, 2, \ldots,$ has the following $\mathcal{F}$-Gabriel-Roiter measure: Since the sequence of Gabriel-Roiter measures of $\mathcal{D}$ is closely related to the Fibonacci sequence, we call it the $\mathcal{F}$-Gabriel-Roiter measure. Thus $\mathcal{F} = \{1, 2, 3,\ 5, 6, 7, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 23, 24, 25, \ldots, 3i - \lfloor i\varphi\rfloor, \ldots\}$, with $\varphi = \frac{1+\sqrt{5}}{2}$ being the golden ratio and $\lfloor\ \rfloor$ the floor function. The letter $\mathcal{F}$ also stands for *full*, since the $\mathcal{F}$-Gabriel-Roiter measure is numerically the fullest measure in the central part.

Conjecture 1 *The* $\mathcal{F}$*-Gabriel-Roiter measure is the infinite Gabriel-Roiter measure* J *for the* 3*-Kronecker quiver, such that* $J < I^t$ *for all* $t \in \mathbb{N}$ *and* $I \le J$ *for all other Gabriel-Roiter measures* I*.*

There is another way to define the Gabriel-Roiter measure of a module of finite length, which was given by Ringel in [Ri06]. Here the Gabriel-Roiter measure is defined by induction on the length of the module and will be a rational number instead of a set of numbers. This has the advantage that one sees immediately which Gabriel-Roiter measure is bigger or smaller, since the usual ordering of rational numbers is used. For the zero module 0 the Gabriel-Roiter measure is $\mu(0) = 0$. Given an indecomposable module M of length $|M| > 0$ and assume by induction that $\mu(M')$ is already defined for any proper submodule $M' \subset M$. Then set $\mu(M) = \max \mu(M') + \frac{1}{2^{|M|}}$, where the maximum is taken over all proper submodules $M' \subset M$. There is the following relationship linking the two definitions: Let $I(M)$ be the Gabriel-Roiter measure written as a set, then $\mu(M) = \sum_{i \in I(M)} \frac{1}{2^i}$.

Theorem 28 *The $\mathcal{F}$-Gabriel-Roiter measure as infinite sum is irrational:*

$$\sum_{i=1}^{\infty} \frac{1}{2^{3n - \lfloor n\varphi \rfloor}} \notin \mathbb{Q}$$

PROOF. Note first that $\mathcal{F}(n) = 3n - \lfloor n\varphi \rfloor$ is a strictly increasing sequence and let us first show that the sequence is not of the form $\mathcal{A}(n) = \mathcal{A}(n+p) + s$, where p denotes the period and s is an integer. If it were, then the slope of the sequence would be s/p, which is a rational number, since s, p are integers. This contradicts the fact that the slope of $\mathcal{F}(n)$ is $3 - \varphi$, an irrational number, since $\varphi = (1 + \sqrt{5})/2$ is the Golden Ratio. Hence $\mathcal{F}(n) = 3n - \lfloor n\varphi \rfloor$ is not of the form $\mathcal{A}(n) = \mathcal{A}(n+p) + s$, which implies that the binary expansion of the infinite sum $\sum_{i=1}^{\infty} \frac{1}{2^{3n - \lfloor n\varphi \rfloor}}$ is not eventually periodic. But this infinite sum is just the definition of the Gabriel-Roiter measure, hence it is an irrational number.

Conjecture 2 *The Gabriel-Roiter measure of $M_\infty = \bigcup M_n$ is $\mu(M_\infty) = \mathcal{F}$. Furthermore, M_∞ can be obtained as the direct limit $M \subset \tau M \subset \tau^2 M \subset \ldots \subset \bigcup_{n=1}^{\infty} \tau^n M = M_\infty$, where M is an indecomposable regular module with dimension vector $\textbf{dim}\, M = (1,1)$.*

Let us recall that in contrast to the tame case, there are no non-zero torsion-free divisible modules (see [Ri79]) over a wild hereditary algebra. The following example of an indecomposable divisible module was constructed by O. Kerner and given in F. Lukas ([L91]):

Example 1 *Let $X \neq 0$ be a regular module with $\mathcal{O}(X)$ a regular mono-orbit*[2]*. Then there is a non-zero map $f : X \to \tau^n X$ for some n. Considering the following chain of monomorphisms $X \overset{f}{\hookrightarrow} \tau^n X \overset{\tau^n f}{\hookrightarrow} \tau^{2n} X \overset{\tau^{2n} f}{\hookrightarrow} \tau^{3n} X \hookrightarrow \ldots$, define $M := \bigcup_r \tau^{rn} X$. Kerner and Lukas [L91] have shown that every proper factor of M is a direct sum of preinjective modules and that M is an indecomposable divisible module.*

---

[2] This means that for all R regular and $n \in \mathbb{N}_0$, all non-zero maps in $\mathrm{Hom}(\tau^n X, R)$ are monomorphisms.

We will closer look at the following chains of inclusions:

$$M \subset \tau M \subset \tau^2 M \subset \tau^3 M \subset \tau^4 M \subset \dots$$

If M is an indecomposable regular module over the 3-Kronecker quiver, then for every $i \in \mathbb{N}$, $\tau^i(M)$ is also indecomposable since τ is an equivalence in the category of regular modules. Recall that if an indecomposable module M has no non-zero regular factor modules, then any morphism $M \to R$, for a regular module R is either zero or injective. Since our interest lies on the chain of dimension vectors $\mathcal{D}$, let us start by considering the case of M being an indecomposable module with dimension vector $\mathbf{dim}(M) = \binom{1}{1}$.

Lemma 29 *Let M be an indecomposable regular module over the* 3*-Kronecker quiver with dimension vector $\boldsymbol{dim}(M) = \binom{1}{1}$. Then we have inclusions $M \hookrightarrow \tau(M) \hookrightarrow \tau^2(M) \hookrightarrow \tau^3(M) \hookrightarrow \tau^4(M) \hookrightarrow \dots$.*

PROOF. We need to show that $\dim \operatorname{Hom}(\tau^i(M), \tau^{i+1}(M)) \neq 0$ and that any $f \in \operatorname{Hom}(\tau^i(M), \tau^{i+1}(M))$ is a monomorphism for $i = 0, 1, 2, 3 \dots$. Since τ is an equivalence it is sufficient to show that $\dim \operatorname{Hom}(M, \tau(M)) \neq 0$. Then, since M has no non-trivial regular factor modules, any such f (if it exists) has to be a monomorphism. Using the Auslander-Reiten formula and bilinear form on dimension vectors we have $\langle \binom{1}{1}, \tau\binom{1}{1} \rangle = -\langle \binom{1}{1}, \binom{1}{1} \rangle = -(-1) = 1$ Thus $\dim \operatorname{Hom}(\binom{1}{1}, \tau\binom{1}{1}) - \dim \operatorname{Ext}(\binom{1}{1}, \tau\binom{1}{1}) = 1$, but $\dim \operatorname{Ext}(\binom{1}{1}, \tau\binom{1}{1}) = 1$, so $\dim \operatorname{Hom}(\binom{1}{1}, \tau\binom{1}{1}) = 2 \neq 0$.

This lemma shows that for any indecomposable module M', such that $\mathbf{dim}(M') = \binom{5}{2} = \Phi\binom{1}{1}$, there exists an indecomposable module M, such that $\mathbf{dim}(M) = \binom{1}{1}$ and $M \subset M'$. Note further, since $\operatorname{Ext}(M, M) \neq 0$, because M is not exceptional (there are no regular exceptional modules for the 3-Kronecker quiver), we have $\dim \operatorname{Hom}(\binom{1}{1}, \tau^i\binom{1}{1}) \neq 0$, for $i \geq 0$ (as shown in [K96], section 4.8). We also have the following result:

Lemma 30 *Let N be an indecomposable regular module over the* 3*-Kronecker quiver with dimension vector $\boldsymbol{dim}(N) = \binom{2}{1}$. Then we have inclusions $N \hookrightarrow \tau N \hookrightarrow \tau^2 N \hookrightarrow \tau^3 N \hookrightarrow \tau^4 N \hookrightarrow \dots$.*

Proposition 31 *Let $M(i)$ be indecomposable regular modules over the* 3*-Kronecker quiver with dimension vector $\boldsymbol{dim}(M(i)) = \binom{1}{1}$, but $M(i) \ncong M(j)$, for $i \neq$*

j, $i, j \in \mathbb{N}$. *Then we have inclusions* $M(1) \hookrightarrow \tau(M(2)) \hookrightarrow \tau^2(M(3)) \hookrightarrow \tau^3(M(4)) \hookrightarrow \tau^4(M(5)) \hookrightarrow \ldots$, *and* $\dim \operatorname{Hom}(\tau^i(M(i-1)), \tau^{i+1}(M(i))) = 1$.

PROOF. It is sufficient (τ being an an equivalence) to show that $\dim \operatorname{Hom}(M(1), \tau(M(2))) = 1$. Then, since M has no non-tivial regular factor modules, any such f (if it exists) has to be a monomorphism. Using the Auslander-Reiten formula and bilinear form on dimension vectors we have $\langle \binom{1}{1}(1), \tau\binom{1}{1}(2)\rangle = -\langle \binom{1}{1}(1), \binom{1}{1}(2)\rangle = -(-1) = 1$. Thus $\dim \operatorname{Hom}(\binom{1}{1}(1), \tau\binom{1}{1}(2)) - \dim \operatorname{Ext}(\binom{1}{1}(1), \tau\binom{1}{1}(2)) = 1$, but $\dim \operatorname{Ext}(\binom{1}{1}(1), \tau\binom{1}{1}(2)) = \dim \operatorname{Ext}(\binom{1}{1}(2), \binom{1}{1}(1)) = \dim$ $DHom(\binom{1}{1}(2), \binom{1}{1}(1)) = 0$, since $M(1) \not\cong M(2)$, so $\dim \operatorname{Hom}(\binom{1}{1}, \tau\binom{1}{1}) = 1$.

Note that for the 3-Kronecker quiver, for any indecomposable module N, such that $\mathbf{dim}(N) = \binom{2}{1}$, there exists an indecomposable module M, such that $\mathbf{dim}(M) = \binom{1}{1}$ and $M \subset N$.

Proposition 32 *Consider the* 3*-Kronecker quiver and let* M *be an indecomposable regular module with dimension vector* $\boldsymbol{dim}(M) = \binom{1}{1}$ *and let* N *be an indecomposable regular module with dimension vector* $\boldsymbol{dim}(N) = \binom{2}{1}$*, such that* $M \subset N$. *Then we have inclusions* $M \hookrightarrow N \hookrightarrow \tau M \hookrightarrow \tau N \hookrightarrow \tau^2 M \hookrightarrow \tau^2 N \hookrightarrow \tau^3 M \hookrightarrow \ldots$, *so* $\dim \operatorname{Hom}(\tau^i M, \tau^i N) \neq 0$*,* $\dim \operatorname{Hom}(\tau^j N, \tau^{j+1} M) \neq 0$*, for all* $i, j \in \mathbb{N}$.

PROOF. Let $M \xrightarrow{f} N$ be an injective map from M to N (N has no non-zero regular factor modules). We want to show that $N \overset{g}{\hookrightarrow} \tau M$ exists. Consider the bilinear form on dimension vectors: Using the Auslander-Reiten formula we get $\langle \binom{2}{1}, \tau\binom{1}{1}\rangle = -\langle \binom{1}{1}, \binom{2}{1}\rangle = -(3-3) = 0$. But $\dim \operatorname{Hom}(M, N) \neq 0$, so $\dim \operatorname{Ext}(M, N) \neq 0$, and so $\dim \operatorname{Hom}(N, \tau M) \neq 0$. Since the Auslander-Reiten translate τ defines an equivalence on the category of regular modules and is right exact, since we are in the hereditary situation, we have the following embeddings for $i, j \in \mathbb{N}$: $\tau^i M \overset{\tau^i f}{\hookrightarrow} \tau^i N$ and $\tau^i N \overset{\tau^j g}{\hookrightarrow} \tau^{j+1} M$. Of course, $\tau^i N$ and $\tau^{j+1} M$ have no non-zero regular factor modules.

References

[CB98] W. Crawley-Boevey: *Infinite-dimensional modules in the representation theory of finite-dimensional algebras.* Canadian Math. Soc. Conf. Proc. 23, 1998, 29-54.

[G73] P. Gabriel: *Indecomposable Representations II.* Symposia Mathematica, Vol. XI, Academic Press, London/New York, 1973, 81-104.

[G81] P. Gabriel: *The universal cover of a representation-finite algebra*, in Representations of Algebras (Puebla, 1980), Springer Lecture Notes in Math., vol. 903, 1981, 68-105.

[K96] O. Kerner: *Representations of wild quivers.* CMS Conf. Proc. 19, 1996, 65-107.

[L91] F. Lukas: *Infinite dimensional modules over wild hereditary algebras.* J. London Math. Soc. 44, 1991, 401-419.

[Ri98] C.M. Ringel: *Exceptional Modules are Tree Modules.* Lin. Alg. Appl. 275-276, 1998, 471-493.

[Ri05] C.M. Ringel: *The Gabriel-Roiter Measure.* Bull. Sci. math. 129, 2005, 726-748.

[Ri06] C.M. Ringel: *Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure.* In: Trends in Representation Theory of Algebras and Related Topics. (Proceedings Queretaro Workshop ICRA 2004, Mexico). Edited by de la Pena and Bautista. Contemporary Math. 406. Amer.Math.Soc., 2006, 105-135.

[Ri79] C.M. Ringel: *Infinite dimensional representations of finite dimensional hereditary algebras.* Symposia Math. 23, 1979, 321-412.

[Ri76] C.M. Ringel: *Representations of K-species and bimodules.* J. Algebra 41, 1976, 269-302.