1. Questions on §1.1 and §1.2.

Question 1.1. Let R be a ring. We revise some details of the Artin-Wedderburn theorem.

(1) Prove *Schur's lemma*, which states that if M and N are simple R-modules then $\mathrm{Hom}_R(N, M) \neq 0$ if and only if $N \cong M$, and furthermore, $\mathrm{End}_R(M)$ is a division ring.

(2) Let $L_1, \ldots, L_n$ be R-modules and let $M = L_1 \oplus \cdots \oplus L_n$.

  (a) For each $i = 1, \ldots, n$ define R-module homomorphisms $\iota_i \colon L_i \to M$ and $\pi_i \colon M \to L_i$ such that $\pi_i \iota_i = \mathrm{id}_{L_i}$, $\pi_j \iota_j = 0$ for $j \neq i$ and $\sum_i \iota_i \pi_i = \mathrm{id}_M$.

  (b) If $L_1 = \cdots = L_n$, from now on denoted L, find a ring isomorphism $\mathrm{End}_R(M) \cong M_n(\mathrm{End}_R(L))$.

  (c) If $\mathrm{Hom}_R(L_i, L_j) = 0$ for $i \neq j$ find a ring isomorphism $\mathrm{End}_R(M) \cong \prod_{i=1}^n \mathrm{End}_R(L_i)$.

(3) Prove that the endomorphism ring of a finite direct sum of simple modules is a product of matrix rings over division rings. Rewrite $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})$.

Question 1.2. Let K be a commutative ring, R be a K-algebra and M be a left R-module. Consider

$$\mathrm{Hom}_R(M, -) \colon R-\mathsf{Mod} \to K-\mathsf{Mod}, \quad \mathrm{Hom}_R(-, M) \colon (R-\mathsf{Mod})^{\mathsf{op}} \to K-\mathsf{Mod}$$

(1) Explain how $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, M)$ are functors (define images on objects and morphisms).

(2) Prove that if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $R-\mathsf{Mod}$ then

$$0 \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(M, Z), \quad 0 \to \mathrm{Hom}_R(Z, M) \to \mathrm{Hom}_R(Y, M) \to \mathrm{Hom}_R(X, M)$$

are exact sequences of K-modules.

Question 1.3. Let K be a field and consider the subring of the matrix ring $M_3(K)$ defined by

$$R = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & 0 & \varepsilon \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon \in K \right\}.$$

(1) Write down a ring homomorphism $R \to K^3$ and hence find an ideal I of R such that $R/I \cong K^3$.

(2) Define left ideals J_1, J_2 and J_3 of R as follows. J_1 consists of matrices with top-left entry $\alpha = 0$, J_2 consists of those with middle entry $\gamma = 0$, and J_3 consists of those with bottom-right entry $\varepsilon = 0$.

  (a) Prove that any non-trivial left ideal of R must be contained in at least one of J_1, J_2 or J_3.

  (b) Prove that $J_2 \cap J_3$ contains a non-trivial left ideal, and hence explain why R is not a semisimple ring. Write down two semisimple K-algebras with that are 5-dimensional. Are there any others? What about when K is algebraically closed? What about when $K = \mathbb{R}$?

Question 1.4. Let K be a field, $\delta \in K$ and consider the Temperley-Lieb algebra $TL_n(\delta)$. Recall the elements $u_i \in TL_n(\delta)$ defined by connecting i and $i+1$ on the left, connecting i and $i+1$ on the right, and connected each $j \neq i, i+1$ on the left with the j on the right.

(1) Prove that if $i, j = 1, \ldots, n$ then $u_i u_i = \delta u_i$, $u_i u_{i\pm1} u_i = u_i$, $u_i u_j = u_j u_i$ for $|i - j| > 1$. Prove that

$$u_4 u_1 u_4 u_3 u_2 u_4 u_1 u_4 u_2 = \delta^2 u_1 u_2 u_4.$$

(2) A *word in* $u_1, \ldots, u_k$ is a product $w = u_{i(1)} \ldots u_{i(d)} \in TL_n(\delta)$ where $1 \leq i(1), \ldots, i(d) \leq k$. Prove, by induction on $c > 0$, that if w is a word in $u_1, \ldots, u_k$ where u_k occurs c-times, then we have $w = \delta^m w' u_k w''$ for some $m \geq 0$ and some words w' and w'' in $u_1, \ldots, u_{k-1}$.

Question 1.5. Let R be the subset of the power set $P(\mathbb{R})$ of $\mathbb{R}$ consisting of the empty set together with finite unions of open intervals (a, b) with $-\infty < a < b < \infty$. Consider the binary operations or addition and multiplication to be the symmetric difference and intersection, respectively. Prove that R is not a catalgebra.