8. Questions on §1.10 and §1.11

Question 8.1. Let R be a ring and let M be a uniserial R-module.

(1) Prove that M is indecomposable and that any submodule or quotient of M is uniserial.
(2) Prove that if M is finitely generated over R then M is cyclic.
(3) Find an example of R and M such that $M \supseteq \mathrm{rad}(M) \supseteq \mathrm{rad}^2(M) \supseteq \cdots$ is not a composition series.

Question 8.2. Let K be a field, let R be a finite-dimensional hereditary K-algebra, and let M and N be R-modules which are finite-dimensional over K.

(1) Let N be projective. Prove that if M is a submodule of N then M is projective. Prove that if $\theta \in \mathrm{Hom}_R(M, N)$ then $M \cong \mathrm{im}(\theta) \oplus \ker(\theta)$.
(2) Let M be indecomposable and nonprojective. Prove that $\mathrm{Hom}_R(M, R) = 0$ and that $\nu(M) = 0$ where ν is the Nakayama functor. State and prove a similar result about indecomposable noninjectives.
(3) Use the Krull–Remak–Schmidt property of finite-length R-modules to describe $\nu(M)$.

Question 8.3. Let K be a field. In this question you are asked to find examples of finite-dimensional algebras that satisfy one property and not the other. In each case, justify why your example satisfies the required property, and justify why it fails the other property.

(1) Find an example of a Frobenius algebra that is no symmetric.
(2) Find an example of a self-injective algebra that is not Frobenius.
(3) Find an example of a QF-3 algebra that is not Nakayama and not self-injective.
(4) Find an example of a finite-dimensional algebra that is not QF-3.