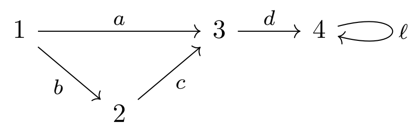
3. Questions on §1.3 and §1.4.

Question 3.1. Let K be a field and let Q be the quiver



(1) Let $n \in \mathbb{N} \cup \{\infty\}$, let $\ell^0 = e_4$, let $\ell^\infty = 0$ and consider the ideal $H = \langle cb, da, \ell^n \rangle$ in KQ. Decide, for which n, H is not admissible, and in each case explain why.

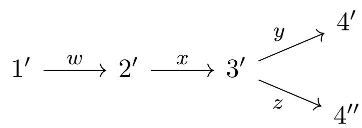
(2) Choose n minimal such that H is admissible. Calculate the representations of Q corresponding to the projective modules $P[1]$, $P[2]$, $P[3]$ and $P[4]$ of the K-algebra KQ/H.

(3) Prove that $I = \langle \ell da - dcb, \ell^2(\ell^2 - e_4) \rangle$ is not admissible. Decide whether of not the K-algebra KQ/I is finite-dimensional, and explain your decision.

(4) Let $J = \langle cb - a, (\ell - \mu e_4)(\ell - \eta e_4) \rangle$ where $\mu, \eta \in K$ and $\mu \neq \eta$. Explain why J is not admissible.

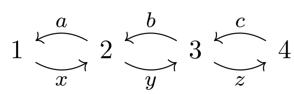
(5) Define elements $f, g \in KQ$ and a new quiver Q' by

$$f = \tfrac{1}{\eta - \mu}(\ell - \mu e_4), \quad g = \tfrac{1}{\mu - \eta}(\ell - \eta e_4),$$



Find a K-algebra isomorphism $\theta \colon KQ' \to KQ/J$ such that $\theta(e_{4'}) = f$ and $\theta(e_{4''}) = g$.

Question 3.2. Let K be a field, and define a quiver Q and an ideal I in KQ by



$$I = \langle xa - by, yb - cz, zc \rangle$$

(1) Using the Diamond Lemma, prove that the K-algebra $A = KQ/I$ has a basis defined by the paths

$$e_1,\ e_2,\ e_3,\ e_4,\ a,\ b,\ c,\ x,\ y,\ z,\ ab,\ bc,\ zy,\ yx,\ ax,\ by,\ cz,$$
$$abc,\ aby,\ bcz,\ byx,\ czy,\ zyx,\ abcz,\ abyx,\ bczy,\ czyx,\ abczy,\ bczyx,\ abczyx.$$

Hint: describe the length-lexicographic ordering defined by declaring that $4 > 3 > 2 > 1$ and $a < b < c < z < y < x$. Prove that a path in Q is irreducible if and only if it is among those above.

(2) Explain why I is admissible. Describe the projective modules $P[1]$ and $P[4]$ as representations of Q.

(3) Find a quiver P with 4 vertices, and find an ideal J of KQ, such that $I \subseteq J$ and such that KQ/J is isomorphic to the preprojective algebra $\Pi(P)$.

(4) Guess a basis for $\Pi(P)$. Write down a module over KQ/I which is not a module over $\Pi(P)$.

(5) Find a vertex v in P such that $e_v \Pi(P) e_v \cong K\langle s, t \rangle / \langle s + t, s^2, t^3 \rangle$. Hint: use a Theorem in the notes.