4. Questions on §1.5 and §1.6.

Question 4.1. Let R be a ring, M be an R-module and $E = \mathrm{End}_R(M)$, the endomorphism ring.

(1) Prove that if $f^2 = f \in E$ then $M = \mathrm{im}(f) \oplus \ker(f)$.
(2) Prove that M is indecomposable if and only if the only idempotents in E and 0 and 1.
(3) Give an example of a ring R and an indecomposable R-module M such that the endomorphism ring $\mathrm{End}_R(M)$ is not local. Explain why you could not have chosen M to have finite length.

Question 4.2. Let R be a ring and $S = M_n(R)$, the ring of $n \times n$ matrices with entries in R.

(1) For $h, k = 1, \ldots, n$ and $r \in R$ define $rE_{hk} = (e_{ij}) \in S$ by $e_{hk} = r$ and $e_{ij} = 0$ for $i \neq h$ or $j \neq k$.
    (a) Prove that $r_{bc}E_{ad} = E_{ab}sE_{cd}$ for $1 \leq a, b, c, d \leq n$ for all $s = (r_{ij}) \in S$.
    (b) Hence, or otherwise, prove that $s(qE_{ij}) = \sum_{h=1}^{n} r_{ij}qE_{hj}$ for any $s = (r_{ij}) \in S$ and any $q \in R$.
(2) Use (1a) to prove that if J is an ideal of S then $I = \{r_{11} \colon (r_{ij}) \in J\}$ in an ideal of R and $J = M_n(I)$.
(3) Use (1b) to prove that, for any $s = (r_{ij}) \in S$, if $1_R - rq \in R$ has left inverse u then

$$(1_S - \textstyle\sum_{j \neq h=1}^{n} r_{hi}qE_{hj})(1_S - (1_R - u)E_{jj})$$

is a left inverse of $1_S - s(qE_{ij}) \in S$.
(4) Using a Theorem from your notes, prove that $M_n(\mathrm{rad}(R)) = \mathrm{rad}(M_n(R))$.

Question 4.3. Let D be a division ring. Let $T_2(D)$ be the set of matrices $(\lambda_{ij}) \in M_2(D)$ with $\lambda_{21} = 0$. Let J be the set of matrices $(\lambda_{ij}) \in T_2(D)$ with $\lambda_{11} = 0 = \lambda_{22}$. Prove that $J = \mathrm{rad}(T_2(D))$. Given a subring S of a ring R, it true that $\mathrm{rad}(S) \subseteq \mathrm{rad}(R)$? Explain your answer.

Question 4.4. Let I be an ideal of a ring R such that $I \subseteq \mathrm{rad}(R)$.

(1) Prove that $\mathrm{rad}(R/I) = \mathrm{rad}(R)/I$. Hence determine $\mathrm{rad}(R/\mathrm{rad}(R))$.
(2) Prove that if $f \colon M \to N$ is a homomorphism of R-modules such that N is finitely generated and such that the induced map $M/IM \to N/IN$ is surjective, prove that f must have been surjective.