5. Questions on §1.6 and §1.7.

Question 5.1. Let $d \in \mathbb{Z}$ with $d > 1$. Explain how to construct a composition series for the $\mathbb{Z}$-module $\mathbb{Z}/d\mathbb{Z}$, and hence compute $\text{length}(\mathbb{Z}/d\mathbb{Z})$. Construct a $\mathbb{Z}$-module of length d.

Let R be a ring. We will prove the Hopkins–Levitzki theorem. There are many steps.

Question 5.2. Let M be an R-module, N a submodule, and given a set I and submodules M_i $(i \in I)$ of M let $L_i = M_i + N$, $K_i = L_i/N$ and $J_i = M_i \cap N$ for each $i \in \mathbb{N}$. **Prove 2 of the following statements**.

(1) If M is artinian then N and M/N are artinian.
(2) Assuming that $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is a descending chain, if $J_j = J_{j+1} = \dots$ and $K_k = K_{k+1} = \dots$ then $M_m = M_{m+1} = \dots$ where $m = j + k$.
(3) If N and M/N are artinian then M is artinian. Explain how to change your argument so far to show M is noetherian iff N and M/N are noetherian.
(4) Assuming $0 = M_0 \subseteq \cdots \subseteq M_n = M$ is a composition series for M, if $i < n$ then exactly one of the quotients K_{i+1}/K_i and J_{i+1}/J_i is 0, and the other is simple.
(5) If $\text{length}(M) < \infty$ then $\text{length}(N) < \infty$ and $\text{length}(M/N) \leq \text{length}(M) - \text{length}(N)$.

Question 5.3. Let M be an R-module and N a submodule. **Prove 2 of the following statements**.

(1) If $\text{length}(N) < \infty$ and $\text{length}(M/N) < \infty$ then $\text{length}(M) \leq \text{length}(M/N) + \text{length}(N)$.
(2) If $\text{length}(M) < \infty$ then M is both artinian and noetherian. Hint: induction on the length.
(3) If $0 = M_0 \subseteq \cdots \subseteq M_n = N$ is a composition series for N, and if M is artinian, then either M/N is simple or there exists a proper submodule M_{n+1} of M such that M_{n+1}/M_n is simple.
(4) If M is both noetherian and artinian then $\text{length}(M) < \infty$.
(5) If N is maximal then a simple submodule of M either lies in N or is a complement to N.

Question 5.4. Let R be *semiprimary*, meaning that $J = \text{rad}(R)$ is nilpotent and $S = R/J$ is semisimple.

(1) Prove that, for each $i \geq 0$, for each $i \geq 0$, the quotient $J^i M/J^{i+1}M$ is isomorphic to a direct sum of simple S-modules. Hint: consider the characterisation of semisimple rings.
(2) Prove that if M is noetherian or artinian then the direct sum above must always be finite, and deduce, using Question 5.3(1), that $\text{length}(J^i M) < \infty$ for each i.
(3) Prove that an R-module is noetherian if and only if it is artinian if and only if it has finite length.

Using the facts above, you may now conclude that artinian rings are the same thing as noetherian semiprimary rings. This is remarkable: although artinian modules and noetherian modules are distinct classes of modules, every artinian ring is automatically a noetherian ring.