6. Questions on §1.7 and §1.8

Question 6.1. Let K be a field and consider the K-subspace of $M_3(K)$ defined by

$$\Lambda := \left\{ \begin{pmatrix} \alpha & \gamma & \varepsilon \\ 0 & \beta & \delta \\ 0 & 0 & \alpha \end{pmatrix} : \alpha, \beta, \gamma, \delta, \epsilon \in K \right\}.$$

(1) Prove that $J := \mathrm{rad}(\Lambda)$ consists of the strictly upper-triangular matrices, so where $\alpha = \beta = 0$.

(2) Prove that the set of idempotents $e^2 = e \in \Lambda$ that are non-trivial (so not $0, 1 \in \Lambda$) is given by

$$E := \left\{ \begin{pmatrix} 0 & \gamma & \gamma\delta \\ 0 & 1 & \delta \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \gamma' & -\gamma'\delta' \\ 0 & 0 & \delta' \\ 0 & 0 & 1 \end{pmatrix} : \gamma, \delta, \gamma', \delta' \in K \right\}.$$

(3) Write the set $\{(e, f) \in E \colon ef = fe = 0\}$, of pairs of orthogonal idempotents, as explicitly as possible.

(4) Choose $(e, f) \in E$ such that $e + f = 1$, explain why they induce a K-basis of Λ/J, prove that Λ is basic and split, and find a semisimple subalgebra S and an ideal N such that $\Lambda = S \oplus N$.

(5) Find bases for the K-vector spaces eJe/eJ^2e, eJf/eJ^2f, fJe/fJ^2e and fJf/fJ^2f.

(6) Use the proof of Gabriel's less famous theorem about quivers to find a quiver Q and an admissible ideal I such that $\Lambda \cong KQ/I$. Calculate the minimal $m > 0$ with $(KQ_+)^m \subseteq I$.

(7) Describe, as representations of Q, the finite-dimensional Λ-modules that are either projective, injective, or simple. Compute the image of each projective under the Nakayama functor.

Question 6.2. Consider the subrings Λ and Γ of $M_2(\mathbb{R})$ defined by

$$\Lambda := \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha \in \mathbb{Q}, \beta, \gamma \in \mathbb{R} \right\}, \quad \Gamma := \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix} = \left\{ \begin{pmatrix} \delta & \epsilon \\ 0 & \zeta \end{pmatrix} : \delta, \epsilon \in \mathbb{R}, \zeta \in \mathbb{Q} \right\}.$$

(1) Compute $\mathrm{rad}(\Lambda)$ and $\mathrm{rad}(\Gamma)$. Explain why both rings are semiprimary.

(2) You are told: one of Λ or Γ is left artinian but not right artinian, and the other is right artinian ring but not left artinian. Using part (1) above, and parts (1) and (3) of Question 5.2, decide which is which. Explain why similar statements are true after replacing *artinian* with *noetherian*.

(3) Find semisimple subrings $S \subseteq \Lambda$ and $T \subseteq \Gamma$, and nilpotent ideals $I \subseteq \Lambda$ and $J \subseteq \Gamma$, such that $\Lambda = S \oplus I$ and $\Gamma = T \oplus J$. Explain why both of these $\mathbb{Q}$-algebras are basic, and why neither is split.