7. Questions on §1.8 and §1.9

Question 7.1. Let R be a ring, $\mathcal{C}$ be a module class in $R-\mathsf{Mod}$, $C \in \mathcal{C}$, $X \in R-\mathsf{Mod}$ and $\varphi \in \mathrm{Hom}_R(C, X)$.

(1) Prove that if φ and $\varphi' : C' \to X$ are $\mathcal{C}$-covers then there is an isomorphism $C' \cong C$.

(2) Assuming $\mathcal{C}$ is the category $R-\mathsf{Proj}$ of projective R-modules, prove that φ is a $\mathcal{C}$-cover if and only if φ is surjective and $A + \ker(\varphi) \neq C$ for any proper submodule $A \subsetneq C$.

(3) Assuming R is artinian and $\mathcal{C} = R-\mathsf{Proj}$, prove that φ is a $\mathcal{C}$-cover if and only if the induced map $\overline{\varphi} : \mathrm{top}(C) \to \mathrm{top}(X)$, defined by $\overline{\varphi}(c + JC) = \varphi(c) + JX$ where $J = \mathrm{rad}(R)$, is an isomorphism.

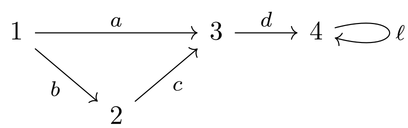
Question 7.2. Let K be a field, R be a finite-dimensional K-algebra, $D = \mathrm{Hom}_{K-\mathsf{Mod}}(-, K)$ and M be a left R-module such that $\dim_K(M) < \infty$.

(1) Prove that if $\varphi : P \to M$ is a projective cover of M then $D(\varphi)$ is an injective envelope of $D(M)$.

(2) Given a minimal projective resolution of M, construct a minimal injective resolution of $D(M)$, and justify why your construction is minimal.

(3) Prove that $\mathrm{proj.dim}({}_R M) = \mathrm{inj.dim}(D(M)_R)$ and that $\mathrm{inj.dim}({}_R M) = \mathrm{proj.dim}(D(M)_R)$.

Question 7.3. Let $\Lambda = KQ/I$ where $I = \langle cb, da, \ell^2 \rangle$ where K is a field and Q is the quiver



(1) Write down the simple Λ-modules and their projective covers as representations of the quiver Q.

(2) Taking kernels, covers (syzygies) compute a minimal projective resolution of each simple.

(3) Write down the indecomposable injective finite-dimensional Λ-modules as representations of Q, and indicate their images under the Nakayama functor ν.

(4) Using Question 7.2, compute a minimal injective resolution for each simple. Write down $\mathrm{gl.dim}(\Lambda)$.

(5) Discuss what you think would have happened if instead one took $I = \langle cb, da, \ell^2 - \ell \rangle$. What do you think would happen to the (projective, injective and global) dimensions?