9. Questions on §2.1 and §2.2

Question 9.1. Let R be a finite-dimensional algebra and let $\mathcal{C}$ be a module class in $R-\mathsf{mod}$. For R-modules M and N define $\mathrm{Hom}^{\mathcal{C}}(M,N)$ to be the set of $\theta \in \mathrm{Hom}_R(M,N)$ such that there exists a module C in $\mathcal{C}$ with $\theta = \beta\alpha$ for some $\alpha \in \mathrm{Hom}_R(M,C)$ and some $\beta \in \mathrm{Hom}_R(C,N)$.

(1) Recall the definition of an *ideal* of $R-\mathsf{mod}$, and prove that $\mathrm{Hom}^{\mathcal{C}}$ is an ideal.

Define a new category $\mathcal{Q} := R-\mathsf{mod}\,/\langle\mathcal{C}\rangle$ with R-modules as objects and

$$\mathrm{Hom}_{\mathcal{Q}}(M,N) := \mathrm{Hom}_R(M,N)/\mathrm{Hom}^{\mathcal{C}}(M,N) = \{[\theta] = \theta + \mathrm{Hom}^{\mathcal{C}}(M,N) \mid \theta \in \mathrm{Hom}_R(M,N)\}.$$

(2) Explain why composition in $\mathcal{Q}$ is well-defined and bilinear over any subring of the centre of R. Construct a functor $\boldsymbol{q}\colon R-\mathsf{mod} \to \mathcal{Q}$ such that $\boldsymbol{q}(C) = 0$ for all C in $\mathcal{C}$.

(3) Let $\theta \in \mathrm{Hom}_R(M,N)$, $\varphi \in \mathrm{Hom}_R(N,M)$. Prove that if $[\varphi\theta] = [1_M]$ then there exists C in $\mathcal{C}$, $\theta' \in \mathrm{Hom}_R(M, N\oplus C)$ and $\varphi' \in \mathrm{Hom}_R(N\oplus C, M)$ with $\varphi'\theta' = 1_M$ and $[\theta\varphi] = [1_N] \Rightarrow [\theta'\varphi'] = [1_{N\oplus C}]$.

(4) Prove that $\boldsymbol{q}(M) \cong \boldsymbol{q}(N)$ if and only if there exist objects C and D in $\mathcal{C}$ such that $M \oplus D \cong N \oplus C$ in $R-\mathsf{mod}$. Hint: use part (3), and then prove that the canonical inclusion $\iota \in \mathrm{Hom}_R(M, M\oplus D)$ gives an isomorphism $[\iota] \in \mathrm{Hom}_{\mathcal{Q}}(M, M\oplus D)$, and then prove $[1_D] = [0]$.

(5) Prove that non-isomorphic non-projective indecomposables in $R-\mathsf{mod}$ cannot be isomorphic in $R-\underline{\mathsf{mod}} = R-\mathsf{mod}\,/\langle R-\mathsf{proj}\rangle$. State and prove a similar result about $R-\overline{\mathsf{mod}} = R-\mathsf{mod}\,/\langle R-\mathsf{inj}\rangle$.

Question 9.2. Let K be a field and let $R = KQ/\langle ca - db\rangle$ where Q is the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ 3 & \xrightarrow[d]{} & 4 \end{array}$$

(1) Prove that $\tau(I[1]) \cong P[1]/S[4]$, $\tau(P[1]/S[4]) \cong \mathrm{rad}(P[1])$ and $\tau(\mathrm{rad}(P[1])) \cong P[4]$.

(2) Prove that $\tau(I[2]) \cong S[3]$, $\tau(S[3]) \cong P[2]$, $\tau(I[3]) \cong S[2]$ and $\tau(S[2]) \cong P[3]$.

(3) Without calculating projective or injective resolutions, prove that $\mathrm{proj.\,dim}(I[1]) \geq 2 \leq \mathrm{inj.\,dim}(P[4])$.

(4) Prove that $\mathrm{rad}(P[1])$ and $P[1]/S[4]$ are indecomposable R-modules that are non-isomorphic in the categories $R-\mathsf{mod}$, $R-\underline{\mathsf{mod}}$ and $R-\overline{\mathsf{mod}}$.

(5) Prove that $\mathrm{Hom}_R(P[1]/s[4], P[i]) = 0$ for each $i = 1,2,3,4$. Deduce that $\underline{\mathrm{Hom}}(P[1]/S[4], I[2]) \neq 0$ and that there is a non-split short exact sequence $0 \to \mathrm{rad}(P[1]) \to M \to I[2] \to 0$ in $R-\mathsf{mod}$.