10. Questions on §2.3 and §2.4

Question 10.1. Let K be a field and let Q be the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{w} & 2 \\ {\scriptstyle z}\big\downarrow & & \big\downarrow{\scriptstyle x} \\ 4 & \xleftarrow[y]{} & 3 \end{array}$$

Let M be the KQ-module given by the representation (V_i, V_a) with $V_1 = V_4 = 0$, $V_2 = V_3 = K$ and $V_x = 1_K$.

(1) Let $L_i = \tau^i M$ for $i = 1, 2, 3$. Describe the representations corresponding to each L_i, proving $L_3 \cong M$.
(2) For each i construct a module N_i with dimension vector $(1, 1, 1, 1)$ and containing L_i as a submodule.
(3) For each i decide which of the modules L_i and N_i are uniserial, and which are not, explaining why.
(4) For each i and j compute $\dim_K(\mathrm{Hom}_{KQ}(L_i, L_j))$ and $\dim_K(\mathrm{Ext}_{KQ}(L_i, L_j))$.

Question 10.2. Let K be a field, and for each pair of (square) $n \times n$ matrices A, B over K, let

$$V[A, B] = \quad K^n \overset{A}{\underset{B}{\rightrightarrows}} K^n$$

a representation of the Kronecker quiver, here denoted Q. Let I_n be the $n \times n$ identity matrix, and let J_n be the $n \times n$ Jordan block matrix with eigenvalue 0 along the diagonal and 1 along the superdiagonal.

(1) Prove that $V[I_2, J_2]$ is indecomposable and that $\tau V[I_2, J_2] \cong V[I_2, J_2]$.
(2) Prove that $V[I_4, J_4]$ is indecomposable and construct a non-split short exact sequence

$$0 \to V[I_2, J_2] \to V[I_4, J_4] \to V[I_2, J_2] \to 0$$

which is not an Auslander–Reiten sequence.

(3) Note that $J_1 = 0$ as a map $K \to K$. Construct an Auslander–Reiten sequence of the form

$$0 \to V[I_2, J_2] \to V[I_3, J_3] \oplus V[I_1, J_1] \to V[I_2, J_2] \to 0.$$

Question 10.3. Let R be a finite-dimensional algebra over a field K and let

$$\alpha : \ 0 \to X \to Y \to Z \to 0$$

be an Auslander–Reiten sequence of finite-dimensional R-modules.

(1) Prove that any Auslander–Reiten sequence that starts with X is isomorphic to α as a 3-term complex
(2) Let $f \in \mathrm{Hom}_R(X, W)$ which is not a split mono. Prove that the pushout of α along f is split.
(3) Prove that the map $Y \to Z$ is an irreducible map, and prove it has an indecomposable kernel.