## 11. Questions on §2.4 and §2.5

**Question 11.1.** Let $R$ be a ring and let $X$ be a left $R$-module.

(1) Prove that if $\operatorname{soc}(X) \nsubseteq \operatorname{rad}(X)$ then there exists a decomposition $X = M \oplus S$ for submodules $S$ and $M$ of $X$ with $S$ simple and $M$ maximal.

Now assume that $R$ is a finite-dimensional algebra, and assume that $X$ is finite-dimensional, indecomposable, projective, injective, and not simple. Write $\iota\colon \operatorname{rad}(X) \to X$ and $\iota'\colon \operatorname{rad}(X)/\operatorname{soc}(X) \to X/\operatorname{soc}(X)$ for the inclusions and $\pi\colon X \to X/\operatorname{soc}(X)$ and $\pi'\colon \operatorname{rad}(X) \to \operatorname{rad}(X)/\operatorname{soc}(X)$ for the projections. Consider

$$\alpha\colon\; 0 \longrightarrow \operatorname{rad}(X) \xrightarrow{\;g = \begin{pmatrix}\iota \\ \pi'\end{pmatrix}\;} X \oplus \operatorname{rad}(X)/\operatorname{soc}(X) \xrightarrow{\;h = \begin{pmatrix}\pi & -\iota'\end{pmatrix}\;} X/\operatorname{soc}(X) \longrightarrow 0$$

(2) Explain why $\operatorname{rad}(X) \ncong X$, why $\iota$ is irreducible and hence prove that $\alpha$ cannot split.
(3) Let $Y$ be a finite-dimensional indecomposable $R$-module and let $f \in \operatorname{Hom}_R(\operatorname{rad}(X), Y)$ be a monomorphism. Prove that either $f$ is a split monomorphism or $f$ factors through $\iota$.
(4) Let $Y$ be a finite-dimensional $R$-module. Prove that any non-split monomorphism $\operatorname{rad}(X) \to Y$ factors through $g$. Hint: apply the Krull–Remak–Schmidt theorem to $Y$.
(5) Prove if $\ell \in \operatorname{End}_R(X \oplus \operatorname{rad}(X)/\operatorname{soc}(X))$ and $\ell g = g$ then there exists $m \in \operatorname{End}_R(X/\operatorname{soc}(X))$ such that $mh = h\ell$. Deduce that $m^n h = h\ell^n$ for all $n > 0$.
(6) Prove that $\alpha$ is an Auslander–Reiten sequence.

**Question 11.2.** Let $R$ be a finite-dimensional algebra and $f \in \operatorname{Hom}_R(X, Y)$ for non-zero finite-dimensional $R$-modules $X$ and $Y$.

(1) Prove that there are no irreducible elements in $\operatorname{rad}^2(X, Y)$.
(2) Let $X$ and $Y$ be indecomposable. Prove that $f$ is irreducible if and only if $f \in \operatorname{rad}(X, Y)\backslash\operatorname{rad}^2(X, Y)$.