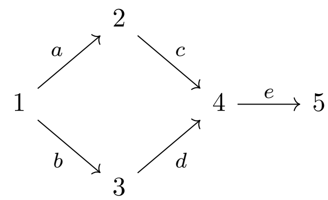
12. Questions on §2.5, §2.6

Question 12.1. Let $R = KQ$ where K is an algebraically closed field and Q is the quiver

(12.1)

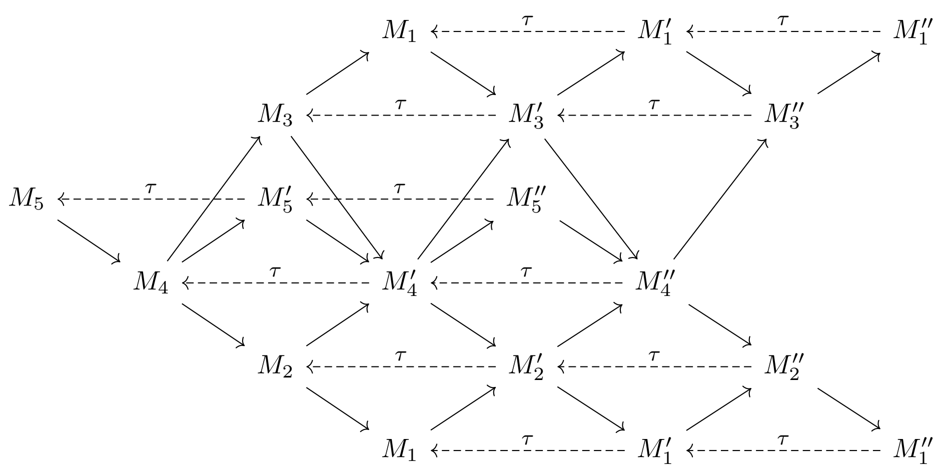


(1) Explain why $\mathrm{rad}(P[i])$ is a direct sum of copies of $P[1], \dots, P[5]$ for each $i = 1, \dots, 5$.
(2) Prove that the inclusion $P[5] \to P[4]$ is a source map and its cokernel $P[4] \to S[4]$ is a sink map.
(3) Compute $\tau^- P[4]$, find sink maps $P[4] \to P[2]$ and $P[4] \to P[3]$, and considering

$$\underline{\dim}(P[4]) + \underline{\dim}(\tau^-(P[4])) - \underline{\dim}(S[4]) - \underline{\dim}(P[2]) - \underline{\dim}(P[3])$$

explain why $\mathrm{irr}(P[4], X) = \mathrm{irr}(X, \tau^- P[4]) = 0$ for any indecomposable $X \neq S[4], P[2], P[3]$.
(4) Run the knitting algorithm for projectives until you have a subquiver Γ' of Γ_R of the form

(12.2)



Write down the dimension vectors for M_i, M'_i and M''_i for each $i = 1, \dots, 5$.
(5) Suppose you are given a subquiver Λ of Γ_R of the form (12.2) and assume that every arrow in Γ_R of the form $M_i \to X$ or of the form $X \to M'_i$ lies in Λ.

For each $i = 1, \dots, 5$ let $m_i = \dim_K(M_i)$ and $m'_i = \dim_K(M'_i)$. Explain why

$$m'_5 = m_4 - m_5, \quad m'_4 = m_3 + m_2 - m_5, \quad m'_3 = m_1 + m_2 - m_5, \quad m'_2 = m_1 + m_3 - m_5.$$

Prove that if $m_1 > m_2 = m_3 > m_4 \geq m_5$ then $m'_1 > m'_2 = m'_3 > m'_4 \geq m'_5$ and hence deduce $m'_5 \geq m_5$ and $m'_i > m_i$ for each $i = 2, \dots, 5$.
(6) Explain why the connected component of Γ_R that contains Γ' from (4) must be infinite.

Question 12.2. Let R be a finite-dimensional algebra.

(1) Let X be an indecomposable R-module. Prove that, if there is a bound on the lengths of paths in the AR quiver starting at X, then X is directing.
(2) Describe the knitting algorithm for the preinjective component of Γ_R. Explain the preparation, construction, iterative step and outcomes.
(3) Starting with injectives, knit Γ_R where R is the commutative square algebra from Question 9.2.