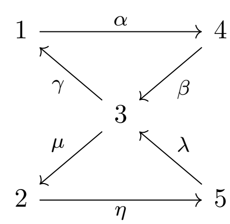
13. Questions on §2.6, §2.7 and §2.8

Question 13.1. Let $R = KQ/I$ where K is a field, $I = \langle \beta\alpha, \gamma\beta, \alpha\gamma, \lambda\eta, \mu\lambda, \eta\mu \rangle$ and Q is the quiver



(1) Describe $P[i]$, $I[i]$, $\mathrm{rad}(P[i])$ and $\mathrm{soc}(I[i])$ for each $i \in Q_0$. Prove $I[3]/\mathrm{soc}(I[3]) \cong \mathrm{rad}(P[1] \oplus P[5])$.
(2) Starting with the source map $\mathrm{rad}(P[3]) \hookrightarrow P[3]$, knit until you get the sink map $I[3] \twoheadrightarrow I[3]/\mathrm{soc}(I[3])$.
(3) Explain why the component you knitted in (2) is the entire Auslander–Reiten quiver, and say what you can about the geometry of the underlying graph.

Question 13.2. Let K be a field. Let R be a graded algebra, so where $R = \bigoplus_{n\in\mathbb{Z}} R_n$ and $R_n R_m \subseteq R_{n+m}$ for each $n, m \in \mathbb{Z}$. Write $R-\mathsf{grmod}$ for the category of finite-dimensional graded left R-modules M, so where $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $R_n M_m \subseteq M_{n+m}$ for each $n, m \in \mathbb{Z}$, and such that $\sum_{n\in\mathbb{Z}} \dim_K(M_n) < \infty$. Let

$$\widehat{R} = \begin{pmatrix} \ddots & \vdots & \vdots & ⋰ \\ \cdots & R_0 & R_{-1} & \cdots \\ \cdots & R_1 & R_0 & \cdots \\ ⋰ & \vdots & \vdots & \ddots \end{pmatrix}$$

Namely, $\widehat{R}$ is the set of matrices $\underline{r} = (r_{ij})$ where the entry in row $i \in \mathbb{Z}$ and column $j \in \mathbb{Z}$ is $r_{ij} \in R_{i-j}$.

(1) Explain why $\widehat{R}$ has the structure of a catalgebra, and describe a small K-category $\mathcal{C}$ such that

$$\widehat{R} \cong \bigoplus_{X,Y\in \mathrm{ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

(2) Characterise the objects in a full subcategory of the category of K-linear functors $\mathcal{C} \to K-\mathsf{Mod}$ which is equivalent to the category $\widehat{R}-\mathsf{mod}$ of finite-dimensional left $\widehat{R}$-modules.
(3) Describe what the equivalence $R-\mathsf{grmod} \to \widehat{R}-\mathsf{mod}$ does on objects and morphisms, and hence describe the K-linear functor $\mathcal{C} \to K-\mathsf{Mod}$ associated to an object of $R-\mathsf{grmod}$.
(4) Explain how any object M in $R-\mathsf{grmod}$ corresponds to a module over an algebra of the form

$$\widehat{R} = \begin{pmatrix} R_0 & 0 & \cdots & 0 & 0 \\ R_1 & R_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{m-n} & R_{m-n-1} & \cdots & R_1 & R_0 \end{pmatrix}$$

for some $n, m \in \mathbb{Z}$ with $n \geq m$. This is the *truncation* of $\widehat{R}$ in degrees $[n, m]$.

Question 13.3. Let $R = KQ/I$ where K is a field, $I = \langle a^2, ab \rangle$ and Q is the quiver

$$2 \xrightarrow{\beta} 1 \circlearrowleft \alpha$$

Let a have degree 1 and b have degree 0.

(1) Compute the truncation T of $\widehat{R}$ in degrees $[-3, 0]$.
(2) By knitting, compute the Auslander–Reiten quiver of T.
(3) Using (2), compute the Auslander–Reiten quiver of R.