Mini-Symposium on Free Boundary Problems

CHAIR: José Francisco Rodrigues

Giovanni Bellettini
Klaus Deckelnick
Irina V. Denisova
Gonzalo Galiano
Harald Garcke
Josephus Hulshof
Régis Monneau
Henrik Shahgholian
José Miguel Urbano

Giovanni Bellettini
Università degli Studi di Roma, Tor Vergata
Facet-breaking in motion by crystalline mean curvature in 3D
Tuesday, 11.30 – 12.00, Room C

Motion by crystalline mean curvature in three dimensions is the gradient flow of an energy functional of the form

$$\int_{\partial E} \psi(\nu) d\mathcal{H}^2,$$

where ν is the outward unit vector to the Lipschitz boundary ∂E and $\psi\colon \mathbb{R}^3 \to [0, +\infty[$ is a convex function, positively homogeneous of degree one and such that $\{\psi \leq 1\}$ is a polyhedron. This evolution law is an example of anisotropic geometric motion; what is of importance here is the presence of corners and flat regions in the unit ball of ψ, which is source of interesting mathematical problems. For instance, we recall that the short time existence theorem (in some class of Lipschitz surfaces which, as we shall see, cannot be reduced to polyhedral surfaces) is still open. Looking at the evolution in relative geometry is the initial point of this talk. Starting from a definition of ψ-"smooth" surface, I will discuss a way to rigorously formulate the geometric evolution problem. Then I will show how to define the crystalline mean curvature of a ψ-smooth surface, and how to uniquely find the initial velocity field on the surface. Regularity of this field on the whole of ∂E and on its facets will be addressed. Finally, I will describe two examples of ψ-smooth evolutions where the phenomena of facet-breaking and facet-curving appear, starting from ψ-smooth polyhedral surfaces. Necessary and sufficient conditions for a facet to break will be discussed.

All results presented here have been obtained in collaboration with M. Novaga (University of Pisa) and M. Paolini (University of Brescia).

Klaus Deckelnick
Sussex University
A phase-field model for diffusion-induced grain-boundary motion
Tuesday, 12.00 – 12.30, Room C

Diffusion-induced grain-boundary motion (DIGM) is observed when a thin polycrystalline film of a metal is exposed to a solute source consisting of a metallic vapour. The solute diffuses down the grain boundaries which separate the crystals and these boundaries start to move leaving a newly created crystal behind the advancing grain.

Cahn, Fife and Penrose have suggested a phase-field model for DIGM, which has the following form:

$$\begin{aligned} \epsilon\phi_t - \epsilon\Delta\phi + \frac{1}{\epsilon}W'(\phi) &\ni u^2 \\ \epsilon u_t - \operatorname{div}\big(D(\phi)\nabla u\big) &= 0 \end{aligned}$$

together with appropriate intial and boundary conditions. Here, ϕ is an order paramater determining the phases $\phi = -1$ and $\phi = +1$ of the polycrystalline material; the boundary region between them corresponds to $-1 < \phi < +1$. $W(\phi)$ denotes the double obstacle potential. Furthermore, u is the concentration of the solute and $D(\phi) = 1 - \phi^2$. As a result, $D(\phi)$ is positive only in the boundary region so that the concentration equation is becomes degenerate.

We present an existence and uniqueness result for the above problem in two space dimensions and discuss a numerical method for approximating its solution.

This is joint work with C. M. Elliott (Sussex) and V. Styles (Oxford Brookes).

Irina V. Denisova
Russian Academy of Sciences, St.Petersburg
Evolution of a closed interface between two liquids of different type
Tuesday, 12.30 – 13.00, Room C

We study a free boundary problem governing the motion of two immiscible viscous capillary fluids. The fluids occupy the whole space $\mathbb{R}^3$ but one of them should have a finite volume. Every liquid may be both of types: compressible and incompressible.

Local (with respect to time) unique solvability of the problem is obtained in the Sobolev-Slobodetskiĭ spaces. First, one considers a model linear problem in $\mathbb{R}^3$ with plane interface between the liquids. One finds an explicit solution of the problem and estimates it in the exponentially weighted spaces. Next, we constract a solution of a linear problem with closed interface in any finite time interval and we make a passage to the ordinary Sobolev-Slobodetskiĭ spaces. Finally, by successive approximations, we prove the solvability of the nonlinear problem in a sufficiently small time interval.

Some restrictions on the fluid viscosities appear in the case when at list one of the liquids is compressible.

Gonzalo Galiano
Universidad de Oviedo
Applications of an energy method to some systems of PDE's involving free boundaries
Thursday, 12.30 – 13.00, Room C

The occurrence of free boundaries in problems formulated in terms of PDE's has been an important topic of research during the last fifty years. Both the broad field of applications in which they arise and the intrinsic mathematical interest of the phenomenum have been the. . . for these investigations. Several methods to identify and localize such free boundaries have been proposed, among them those based in comparison principles. We present here a method based on local energy estimates which allows us to deal with a great variety of problems. These problems may have a very general form, where *generality* here stands for:

- No global information, like boundary conditions or boundedness of the domain, is needed.
- No monotonicity assumption on the nonlinearities is needed, as the comparison principle is not invoked.
- Coefficients may depend on space and time variables and only a weak regularity is assumed.
- No restriction on the space dimension is assumed.

The method was introduced by S. N. Antontsev in the early 80's and developed since then by a number of authors, among whom J. I. Díaz, L. Véron, S. Shamrev and F. Bernis.

In this talk, we present the general ideas of the method and apply it to a number of problems arising in the continuum mechanics context, such as the Boussinesq system of hydrodynamics, the drift-diffusion model for semiconductors, and others.

Harald Garcke
Universität Bonn
Phase boundaries in alloys with elastic misfit
Wednesday, 11.30 – 12.00, Room C

Many problems involving free boundaries arise in the theory of phase transitions. In this context the interfaces separating two or more phases are the boundaries that have to be determined as part of the problem. In my talk I will consider phase transformations in alloys with kinetics driven by mass transport and stress. Different phases in such systems are characterised by certain distinct mixtures of the alloy components.

To describe the dynamics, we study a Cahn-Hilliard system taking elastic effects into account. Existence and uniqueness results for the resulting elliptic parabolic system are given. Furthermore, we present numerical simulations that demonstrate that the elastically modified Cahn-Hilliard model recovers features observed in experiments.

In the Cahn-Hilliard model phase boundaries are described by a diffuse interface with small positive thickness. In the stationary case we identify the sharp interface free boundary problem that arises when the interfacial thickness tends to zero. In particular we obtain a geometric partition problem generalising variants of isoperimetric problems to situations where elastic interactions cannot be neglected.

Josephus Hulshof
Universiteit Leiden
Some aspects of the thin film equation
Wednesday, 12.00 – 12.30, Room C

In this talk I will discuss solutions of the partial differential equation $u_t + (u^n u_{xxx})_x = 0$. Here subscripts denote partial derivatives, n is parameter, and the unknown is $u(x, t)$. This degenerate fourth order parabolic equations arises in models for thin film flows. It looks deceptively similar to the well known porous media equation, which reads $u_t = (u^n u_x)_x$ and is of second order.

I will concentrate on similarity solutions. The ordinary differential equation for the similarity profile leads, after a transformation which mods out a scaling invariance, to a four dimensional quadratic dynamic system in which we look for heteroclinic orbits connecting two critical points, one of which lies at infinity.

Régis Monneau
CERMICS and École Nationale des Ponts et Chaussées, Marne-la-Vallée
Some regularity results for the obstacle problem
Wednesday, 12.30 – 13.00, Room C

We present some new results in dimension two. Our approach mainly refer to the work of L. A. Caffarelli on the obstacle problem. On one hand improving one result of L. A. Caffarelli–N. M. Rivière, we prove that sufficiently small components have at most two singular tips. On another hand, answering a question asked by D. G. Schaeffer, we also show that generic free boundaries are smooth.

Henrik Shahgholian
Royal Institute of Technology, Stockholm
The impact of monotonicity formulas in regularity of free boundaries
Thursday, 11.30 – 12.00, Room C

In this talk we give a survey on recent developments in the regularity of free boundaries of obstacle type in absence of the obstacle, giving rise to solutions that may change sign. The focus is on two techniques, the monotonicity formulas and global versus local analysis.

Report on a recent work by: L. Caffarelli, L. Karp and H. Shahgholian. Partially supported by the Swedish Natural Science Research Council.

José Miguel Urbano
Universidade de Coimbra
A free boundary problem: contributions from modern analysis
Thursday, 12.00 – 12.30, Room C

We exemplify the role of Free Boundary Problems as an important source of ideas in modern analysis. With the help of a model problem we illustrate the use of analytical, algebraic and geometrical techniques obtaining uniqueness of weak solutions via the use of entropy inequalities, existence through nonlinear semigroup theory, and regularity using a method, called intrinsic scaling, based on interpreting a partial differential equation in a geometry dictated by its own structure.