Asymptotic behavior of beta-integers

Ľubomíra Balková, Jean-Pierre Gazeau, Edita Pelantová

Aspects of Aperiodic Order, Bielefeld

3.7.2008

Outline

- Definition of beta-integers
- Properties of beta-integers
- Parry numbers
- Results on asymptotic behavior
- Known results on asymptotic behavior
- Sketch of the proof
- Open problem

Outline

- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \dots x_0 \bullet x_{-1} \dots, \qquad x_i \in \mathbb{N},$$

is a β-*representation* of x

- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots, \qquad x_i \in \mathbb{N},$$

  is a *β-representation* of x
- *β-expansion* $\langle x \rangle_\beta$ of x = β-representation of x obtained by the greedy algorithm

- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots, \qquad x_i \in \mathbb{N},$$

  is a β-*representation* of x
- β-*expansion* $\langle x \rangle_\beta$ of x = β-representation of x obtained by the greedy algorithm
- coefficients of β-expansions in $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$

- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots, \qquad x_i \in \mathbb{N},$$

  is a β-*representation* of x
- β-*expansion* $\langle x \rangle_\beta$ of x = β-representation of x obtained by the greedy algorithm
- coefficients of β-expansions in $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$
- $\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k x_{k-1} \ldots x_0 \bullet \}$

Outline

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic
- $\mathbb{Z}_\beta$ has no accumulation points

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic
- $\mathbb{Z}_\beta$ has no accumulation points

$$\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$$

with $b_0 = 0$ and $b_{n+1} > b_n$

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic
- $\mathbb{Z}_\beta$ has no accumulation points

$$\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$$

  with $b_0 = 0$ and $b_{n+1} > b_n$
- distances in $\mathbb{Z}_\beta$ bounded by 1

for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic
- $\mathbb{Z}_\beta$ has no accumulation points

$$\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$$

  with $b_0 = 0$ and $b_{n+1} > b_n$
- distances in $\mathbb{Z}_\beta$ bounded by 1
- $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$

Outline

- β is a *Parry number* if the number of distances in $\mathbb{Z}_\beta$ finite

- β is a *Parry number* if the number of distances in $\mathbb{Z}_\beta$ finite
- Pisot numbers $\subset$ Parry numbers $\subset$ Perron numbers

- β is a *Parry number* if the number of distances in $\mathbb{Z}_\beta$ finite
- Pisot numbers $\subset$ Parry numbers $\subset$ Perron numbers
- $\mathbb{Z}_\beta^+$ coded by an infinite word u_β:

$$\begin{array}{rcl} \Delta_0 & \longrightarrow & 0 \\ \Delta_1 & \longrightarrow & 1 \\ & \vdots & \\ \Delta_{m-1} & \longrightarrow & m-1 \end{array}$$

- β is a *Parry number* if the number of distances in $\mathbb{Z}_\beta$ finite
- Pisot numbers $\subset$ Parry numbers $\subset$ Perron numbers
- $\mathbb{Z}_\beta^+$ coded by an infinite word u_β:

$$\begin{array}{ccc} \Delta_0 & \to & 0 \\ \Delta_1 & \to & 1 \\ & \vdots & \\ \Delta_{m-1} & \to & m-1 \end{array}$$

- $u_\beta =$ fixed point of one of 2 possible primitive substitutions (simple and non-simple Parry)

Outline

If φ, $u_\beta = \varphi(u_\beta)$, of the form

$$\begin{array}{rcl}\varphi(0) & = & 0^{t_1}1 \\ \varphi(1) & = & 0^{t_2}2 \\ & \vdots & \\ \varphi(m-2) & = & 0^{t_{m-1}}(m-1) \\ \varphi(m-1) & = & 0^{t_m}\end{array}$$

with $t_i \in \mathbb{N}$ and $t_j t_{j+1} \cdots t_m \prec t_1 t_2 \cdots t_m$ for every $1 < j \leq m$ and $t_m \neq 0$, then

β is a *simple Parry number*

If φ, $u_\beta = \varphi(u_\beta)$, of the form

$$\begin{array}{rcl} \varphi(0) & = & 0^{t_1}1 \\ \varphi(1) & = & 0^{t_2}2 \\ & \vdots & \\ \varphi(m-2) & = & 0^{t_{m-1}}(m-1) \\ \varphi(m-1) & = & 0^{t_m} \end{array}$$

with $t_i \in \mathbb{N}$ and $t_j t_{j+1} \cdots t_m \prec t_1 t_2 \cdots t_m$ for every $1 < j \leq m$ and $t_m \neq 0$, then

β is a *simple Parry number*

Parry polynomial of β:

$$p(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \cdots - t_m$$

Outline

Theorem

If β is a simple Parry number, then

$$c_\beta := \lim_{n \to +\infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m - 1} p'(\beta).$$

Theorem

If β is a simple Parry number, then

$$c_\beta := \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

Theorem

If β is a simple Parry number, then

$$c_\beta := \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

Conjecture

Let β be a simple Parry number. Then, $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded iff β is a Pisot number such that its Parry and minimal polynomial coincide.

Outline

Theorem (Gazeau, Verger-Gaugry)

If β is a simple Parry quadratic unit, then

$$\mathbb{Z}_\beta^+ = \left\{ b_n = c_\beta n + \frac{1-\beta}{\beta^2+\beta} + \frac{\beta-1}{\beta} \left\{ \frac{n+1}{1+\beta} \right\},\ n \in \mathbb{N} \right\},$$

where $c_\beta = \dfrac{\beta^2+1}{\beta^2+\beta}$.

Outline

- Substitution matrix M associated with φ by $M_{ij}=|\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

is primitive

- Substitution matrix M associated with φ by $M_{ij} = |\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

  is primitive
- Perron-Frobenius:
  - M has an algebraically simple eigenvalue $\lambda > |\alpha|$ for any other eigenvalue α of M
  - only to λ corresponds a positive eigenvector

- Substitution matrix M associated with φ by $M_{ij} = |\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

  is primitive
- Perron-Frobenius:
  - M has an algebraically simple eigenvalue $\lambda > |\alpha|$ for any other eigenvalue α of M
  - only to λ corresponds a positive eigenvector
- Characteristic polynomial of M = Parry polynomial

- Substitution matrix M associated with φ by $M_{ij} = |\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

  is primitive
- Perron-Frobenius:
  - M has an algebraically simple eigenvalue $\lambda > |\alpha|$ for any other eigenvalue α of M
  - only to λ corresponds a positive eigenvector
- Characteristic polynomial of M = Parry polynomial
- $(\beta^{m-1}, \dots, \beta, 1)$ = left eigenvector of M associated with β

- Substitution matrix M associated with φ by $M_{ij}=|\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

  is primitive
- Perron-Frobenius:
  - M has an algebraically simple eigenvalue $\lambda > |\alpha|$ for any other eigenvalue α of M
  - only to λ corresponds a positive eigenvector
- Characteristic polynomial of M = Parry polynomial
- $(\beta^{m-1}, \dots, \beta, 1)$ = left eigenvector of M associated with β
- Queffélec: Vector of letter frequencies = left eigenvector $(\rho_0, \rho_1, \dots, \rho_{m-1})$ normalized by $\sum_{i=0}^{m-1} \rho_i = 1$

Outline

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

- $b_n = |u|_0\Delta_0 + |u|_1\Delta_1 + \cdots + |u|_{m-1}\Delta_{m-1}$, where u prefix of u_β of length n

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

- $b_n = |u|_0 \Delta_0 + |u|_1 \Delta_1 + \cdots + |u|_{m-1} \Delta_{m-1}$, where u prefix of u_β of length n
- $c_\beta = \rho_0 \Delta_0 + \rho_1 \Delta_1 + \cdots + \rho_{m-1} \Delta_{m-1}$

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

- $b_n = |u|_0 \Delta_0 + |u|_1 \Delta_1 + \cdots + |u|_{m-1} \Delta_{m-1}$, where u prefix of u_β of length n
- $c_\beta = \rho_0 \Delta_0 + \rho_1 \Delta_1 + \cdots + \rho_{m-1} \Delta_{m-1}$
- $(\rho_0, \rho_1, \ldots, \rho_{m-1}) = \frac{1}{\sum_{i=0}^{m-1} \beta^i} (\beta^{m-1}, \ldots, \beta, 1)$

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

- $b_n = |u|_0\Delta_0 + |u|_1\Delta_1 + \cdots + |u|_{m-1}\Delta_{m-1}$, where u prefix of u_β of length n
- $c_\beta = \rho_0\Delta_0 + \rho_1\Delta_1 + \cdots + \rho_{m-1}\Delta_{m-1}$
- $(\rho_0, \rho_1, \ldots, \rho_{m-1}) = \frac{1}{\sum_{i=0}^{m-1} \beta^i}(\beta^{m-1}, \ldots, \beta, 1)$
- Δ_i known (Thurston)

Theorem

If β is a simple Parry number, then

$$c_\beta \quad := \quad \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

- $b_n = |u|_0\Delta_0 + |u|_1\Delta_1 + \cdots + |u|_{m-1}\Delta_{m-1}$, where u prefix of u_β of length n
- $c_\beta = \rho_0\Delta_0 + \rho_1\Delta_1 + \cdots + \rho_{m-1}\Delta_{m-1}$
- $(\rho_0, \rho_1, \ldots, \rho_{m-1}) = \frac{1}{\sum_{i=0}^{m-1}\beta^i}(\beta^{m-1}, \ldots, \beta, 1)$
- Δ_i known (Thurston)
- adroit manipulation

Outline

- Fabre: $U_i := |\varphi^i(0)|$, then

$$n = \sum_{i=0}^{k} a_i U_i \quad \text{if} \quad \langle b_n \rangle_\beta = a_k \ldots a_1 a_0 \bullet$$

- Fabre: $U_i := |\varphi^i(0)|$, then

$$n = \sum_{i=0}^{k} a_i U_i \quad \text{if} \quad \langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$$

- for any $w \in \{0, 1, \dots, m-1\}^*$

$$(|w|_0, |w|_1, \dots, |w|_{m-1}) M = (|\varphi(w)|_0, |\varphi(w)|_1, \dots, |\varphi(w)|_{m-1})$$

- Fabre: $U_i := |\varphi^i(0)|$, then

$$n = \sum_{i=0}^{k} a_i U_i \quad \text{if} \quad \langle b_n \rangle_\beta = a_k \ldots a_1 a_0 \bullet$$

- for any $w \in \{0, 1, \ldots, m-1\}^*$

$$(|w|_0, |w|_1, \ldots, |w|_{m-1})M = (|\varphi(w)|_0, |\varphi(w)|_1, \ldots, |\varphi(w)|_{m-1})$$

- $U_i = (1, 0, \ldots, 0)M^i(1, 1, \ldots, 1)^T$

Outline

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$
- Parry $=$ minimal polynomial $\Rightarrow$ distinct roots $\Rightarrow$ M diagonalizable

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$
- Parry = minimal polynomial ⇒ distinct roots ⇒ M diagonalizable
- $U_i = (1, 0, \ldots, 0) P^{-1} \begin{pmatrix} \beta^i & 0 & \ldots & 0 \\ 0 & \beta_2^i & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta_m^i \end{pmatrix} P(1, 1, \ldots, 1)^T$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$
- Parry = minimal polynomial ⇒ distinct roots ⇒ M diagonalizable
- $U_i = (1,0,\ldots,0)P^{-1} \begin{pmatrix} \beta^i & 0 & \ldots & 0 \\ 0 & \beta_2^i & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta_m^i \end{pmatrix} P(1,1,\ldots,1)^T$
- $\frac{1}{c_\beta} = (1,0,\ldots,0)P^{-1} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} P(1,1,\ldots,1)^T$

Outline

- if $\langle b_n \rangle_\beta = a_k \ldots a_1 a_0 \bullet$, then

- if $\langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i \right) =$

- if $\langle b_n\rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i\right) =$

$$= (1, 0, \dots, 0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & -z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P(1, 1, \dots, 1)^T,$$

- if $\langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i \right) =$

$$= (1, 0, \dots, 0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & -z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P (1, 1, \dots, 1)^T,$$

where $z_j^{(n)} = \sum_{i=0}^{k} a_i \beta_j^i$

- if $\langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i \right) =$

$$= (1, 0, \dots, 0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & -z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P (1, 1, \dots, 1)^T,$$

  where $z_j^{(n)} = \sum_{i=0}^{k} a_i \beta_j^i$
- β Pisot $\wedge$ Parry polynomial = minimal polynomial $\wedge$ $a_i \in \{0, 1, \dots, \lceil \beta \rceil - 1\} \Rightarrow \left(z_j^{(n)} \right)_{n \in \mathbb{N}}$ is bounded for $j \in \{2, \dots, m\}$

- if $\langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i \right) =$

$$= (1,0,\dots,0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & -z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P (1,1,\dots,1)^T,$$

  where $z_j^{(n)} = \sum_{i=0}^{k} a_i \beta_j^i$
- β Pisot $\wedge$ Parry polynomial $=$ minimal polynomial $\wedge$ $a_i \in \{0,1,\dots,\lceil \beta \rceil - 1\} \Rightarrow \left(z_j^{(n)} \right)_{n \in \mathbb{N}}$ is bounded for $j \in \{2,\dots,m\}$

$$\boxed{\frac{1}{c_\beta} b_n - n = \sum_{j=2}^{m} \frac{-z_j^{(n)}}{p'(\beta_j)} \frac{1-\beta_j^m}{1-\beta_j}}$$

Outline

No idea about asymptotic behavior of β-integers for non-Parry numbers β, No idea about asymptotic behavior of β-integers for non-Parry numbers β, i.e., when $\mathbb{Z}_\beta$ assumes infinitely many distinct distances between neighbors.

No idea about asymptotic behavior of β-integers for non-Parry numbers β, i.e., when $\mathbb{Z}_\beta$ assumes infinitely many distinct distances between neighbors.

Suggestion: Start with the existence of $\lim_{n\to\infty} \frac{b_n}{n}$.