Integer Cohomology of Canonical Projection Tilings

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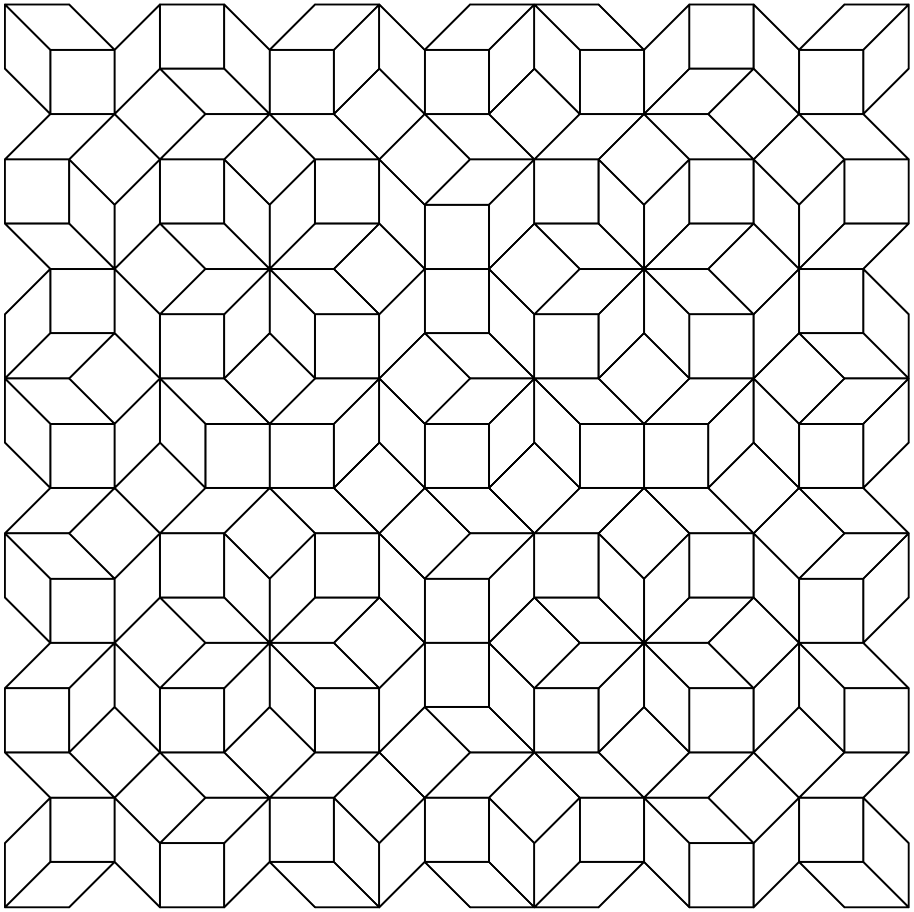
Properties of Tilings



- finite number of local patterns (finite local complexity)
- repetitivity
- well-defined patch frequencies
- local isomorphism (LI classes)

The Hull of a Tiling

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$, of finite local complexity.

We introduce a metric on the set of translates of $\mathcal{T}$:

Two tilings have distance $< \epsilon$, if they agree in a ball of radius $1/\epsilon$ around the origin, up to a translation $< \epsilon$.

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$\Omega_{\mathcal{T}}$ is a compact metric space, on which $\mathbb{R}^d$ acts by translation.

If $\mathcal{T}$ is repetitive, every orbit is dense in $\Omega_{\mathcal{T}}$.
$\Omega_{\mathcal{T}}$ then consists of the LI class of $\mathcal{T}$.

Approximating the Hull by CW-Spaces

We define a sequence of cellular CW-spaces Ω_n approximating Ω.

The d-cells of Ω_0 are the interiors of the tiles; two tile boundaries are identified if they are shared somewhere in the tiling.

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There are natural, continuous cellular mappings $h : \Omega_n \to \Omega_{n-1}$, and induced homomorphisms $h_* : H^*(\Omega_n) \to H^*(\Omega_{n+1})$.

Ω then is the inverse limit $\varprojlim \Omega_n$, consisting of all sequences $\{x_k\}_{k=0}^{\infty}$, with $x_k \in \Omega_k$ and $h(x_k) = x_{k-1}$.

The cohomology of Ω is the direct limit $H^*(\Omega) \cong \varinjlim H^*(\Omega_n)$

Cohomology of Substitution Tilings

The appromimants Ω_n of the hull were introduced by Anderson and Putnam (AP), Ergod. Th. & Dynam. Sys. 18, 509 (1998).

They used a single CW-space Ω' and the mapping $\Omega' \to \Omega'$ induced by substitution, and take the inverse limit of the iterated mapping.

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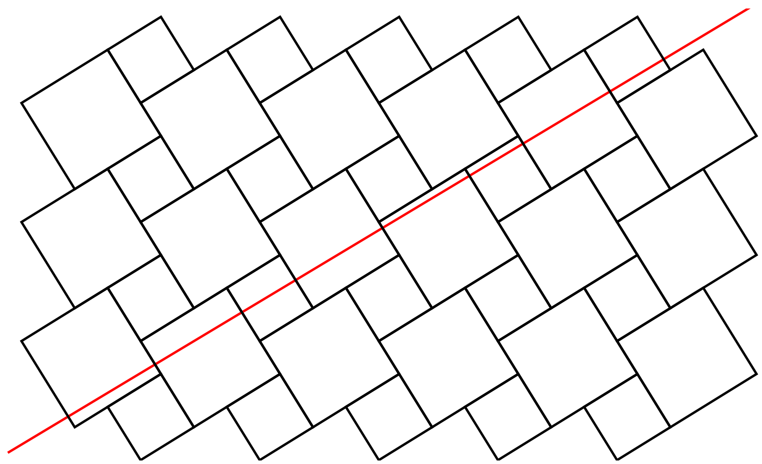
This inverse limit using a single Ω_n is easier to control, but is limited to substitution tilings.

Using a sequence of Ω_n is more general, but the limit is hard to control. However, the approach may be of conceptual interest.

Quasiperiodic Projection Tilings



Irrational sections through a periodic klotz tiling.

We assume polyhedral acceptance domains with rationally oriented faces.

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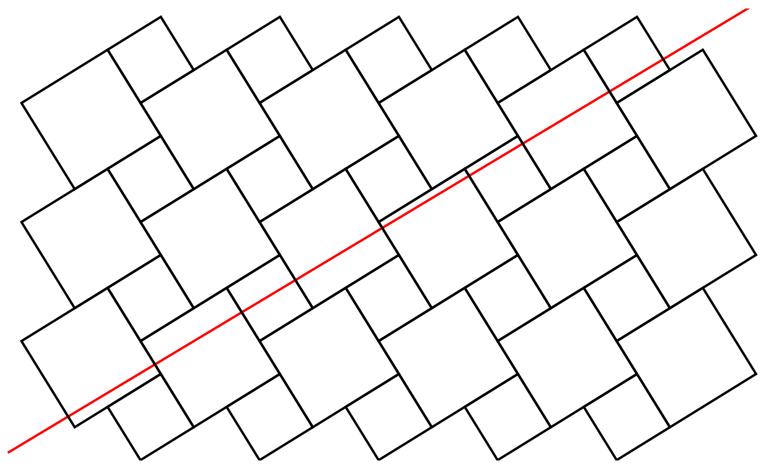

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For canonical projection tilings, Forrest-Hunton-Kellendonk computed cohomology for low co-dimensions in terms of acceptance domains, and claimed the cohomology is free.

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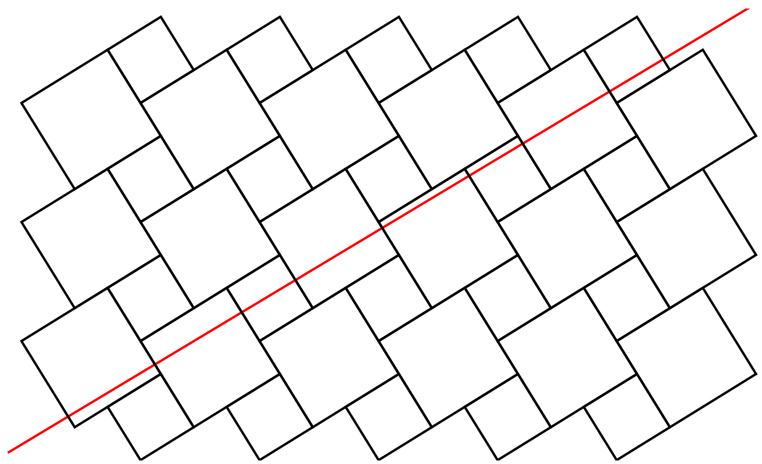

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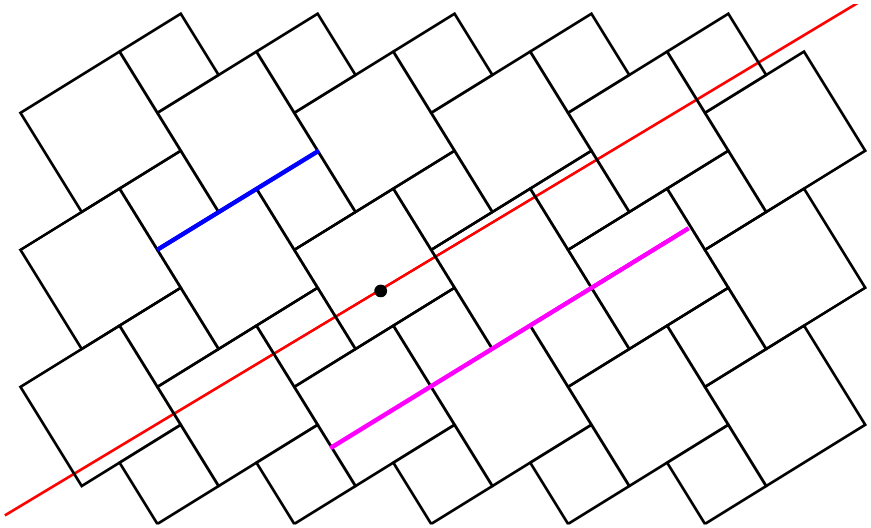
For canonical projection tilings, Forrest-Hunton-Kellendonk computed cohomology for low co-dimensions in terms of acceptance domains, and claimed the cohomology is free.

However, I found a counter-example with torsion ...
This tiling is also substitutional $\longrightarrow$ Anderson-Putnam method!

For co-dimension 2, FHK theory is easily fixed, but co-dimension 3 is much harder...

Kalugin's Approach



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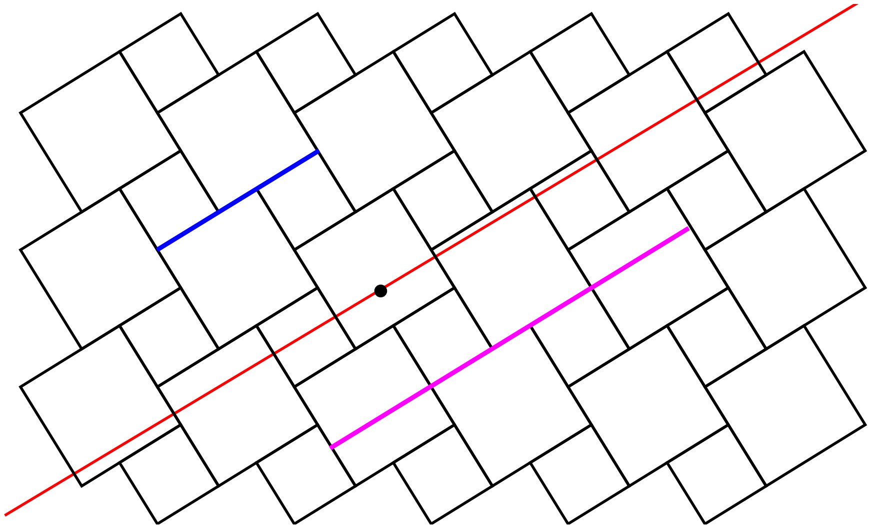

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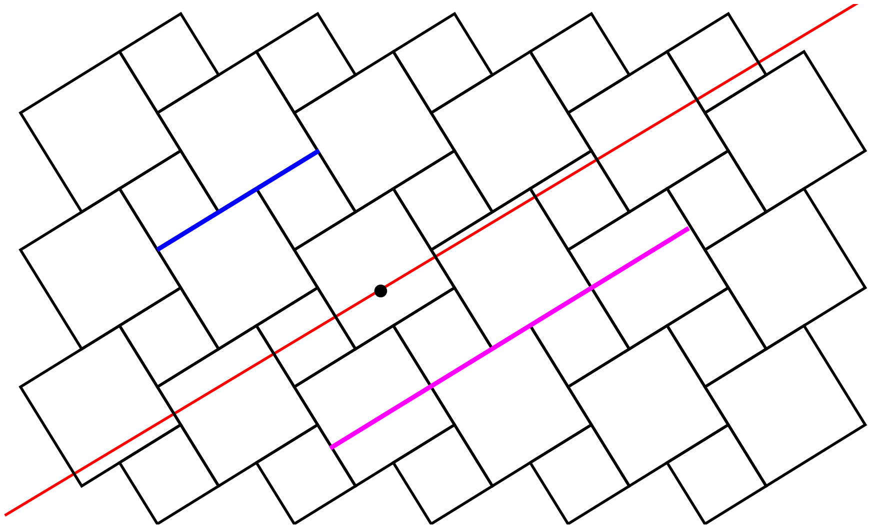

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Cohomology of n-torus cut up along set A_r satisfies

$$\longrightarrow H^k(\Omega_r) \longrightarrow H_{n-k-1}(A_r) \longrightarrow H_{n-k-1}(\mathbb{T}^n) \longrightarrow H^{k+1}(\Omega_r) \longrightarrow$$

P. Kalugin, J. Phys. A: Math. Gen. **38**, 3115 (2005).

Simplifying the Set of Cuts

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For icosahedral tilings, $\tilde{A}$ consists of 4-tori, intersecting in 2-tori and 0-tori.

Kalugins Exact Sequences

Kalugin's long exact sequence can be split; for dimension 3, co-dimension 3 tilings, it reads:

$$0 \longrightarrow S_k \longrightarrow H^k(\Omega) \longrightarrow H_{6-k-1}(A) \xrightarrow{\alpha^{k+1}} H_{6-k-1}(\mathbb{T}^6) \longrightarrow S_{k+1} \longrightarrow 0$$

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0 & \longrightarrow & H_5(\mathbb{T}^6) & \longrightarrow & H^1(\Omega) & \longrightarrow & H_4(A) & \longrightarrow & H_4(\mathbb{T}^6) & \longrightarrow & S_2 & \longrightarrow & 0 \\
0 & \longrightarrow & S_2 & \longrightarrow & H^2(\Omega) & \longrightarrow & H_3(A) & \longrightarrow & H_3(\mathbb{T}^6) & \longrightarrow & S_3 & \longrightarrow & 0 \\
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\end{array}$$

We need to determine $H_*(\mathbb{T}^6)$ (easy), $H_*(A)$ (doable), $S_k = \operatorname{coker} \alpha^k$ (sometimes hard) for different coefficient groups $R = \mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{F}_p$, and derive $H^*(\Omega)$ from that.

Mayer-Vietoris Spectral Sequence

First page $E^1_{k,\ell}$ of Mayer-Vietoris double complex for $H_*(A)$:

$\oplus_{\alpha\in I_2}\Lambda_4\Gamma^\alpha$		
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$\oplus_{\alpha\in I_2}\Lambda_2\Gamma^\alpha \bigoplus \oplus_{\theta\in I_1}\Lambda_2\Gamma^\theta$	$\oplus_{\alpha\in I_2}\oplus_{\theta\in I_1^\alpha}\Lambda_2\Gamma^\theta$	
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$\mathbb{Z}^{L_2} \bigoplus \mathbb{Z}^{L_1} \bigoplus \mathbb{Z}^{L_0}$	$\oplus_{\alpha\in I_2}\mathbb{Z}^{L_1^\alpha} \bigoplus \oplus_{\alpha\in I_2}\mathbb{Z}^{L_0^\alpha} \bigoplus \oplus_{\theta\in I_1}\mathbb{Z}^{L_0^\theta}$	$\oplus_{\alpha\in I_2}\oplus_{\theta\in I_1^\alpha}\mathbb{Z}^{L_0^\theta}$

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As all these groups are free, and $H_0(A) = \mathbb{Z}$, $H_1(A) = \mathbb{Z}^6$.
Only one differential needs to be known explicitly:

$$d^1_{1,2} : \oplus_{\alpha\in I_2} \left(\oplus_{\theta\in I_1^\alpha}\Lambda_2\Gamma^\theta\right) \longrightarrow \left(\oplus_{\alpha\in I_2}\Lambda_2\Gamma^\alpha\right) \oplus \left(\oplus_{\theta\in I_1}\Lambda_2\Gamma^\theta\right)$$

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Torsion only in $E^\infty_{0,\ell}$; Hence, $H_n(A)$ is direct sum of $E^\infty_{k,\ell}$, $n = k + \ell$.

Similar results for other coefficient groups $R = \mathbb{Q}$ or $R = \mathbb{F}_p$.

Homology of A

For any coefficient group $R = \mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{F}_p$, we get:

$$H_0(A;R) = R$$

$$H_1(A;R) = R^6$$

$$H_2(A;R) = \text{coker } d^1_{1,2}(R) \oplus R^f$$

$$H_3(A;R) = (\oplus_{\alpha\in I_2}\Lambda_3\Gamma^\alpha) \oplus (\ker\, d^1_{1,2}(R))$$

$$H_4(A;R) = \oplus_{\alpha\in I_2}\Lambda_3\Gamma^\alpha$$

with $\quad f = -3L_2 - L_1 + \sum_{\alpha\in I_2} L_1^\alpha + 5 + \chi$

and $\quad \chi = L_0 - \sum_{\alpha\in I_2} L_0^\alpha + \sum_{\alpha\in I_2}\sum_{\theta\in I_1^\alpha} L_0^\theta - \sum_{\theta\in I_1} L_0^\theta$

For $R = \mathbb{Z}$, the only potential source of torsion in $H_*(A;\mathbb{Z})$ is $\text{coker } d^1_{1,2}(\mathbb{Z})$.
χ is the Euler characteristic of Ω.

$\mathbf{S_3}$ – Rational Coefficients

Want to show: $S_3(\mathbb{Q}) = \mathrm{coker}\,\alpha^3_{\mathbb{Q}} = \Lambda_3\Gamma \otimes \mathbb{Q} / \langle \Lambda_3\Gamma^\alpha \otimes \mathbb{Q} \rangle$.

Clearly, image of $\alpha^3_{\mathbb{Q}}$ is at least $M = \langle \Lambda_3\Gamma^\alpha \otimes \mathbb{Q} \rangle$.

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To show this is all, consider a simplicial decomposition of $\mathbb{T}^6$, containing A as a subcomplex, and take the chain maps

$$C_3(A) \xrightarrow{i} C_3(\mathbb{T}^6) \xrightarrow{P} \Lambda_3\Gamma \otimes \mathbb{Q}$$

P assigns to each simplex its volume form. P vanishes on boundaries, and induces an isomorphism $H_3(\mathbb{T}^6) \longrightarrow \Lambda_3\Gamma \otimes \mathbb{Q}$.

All cells in A span directions in M, and so the image of $\alpha^3_{\mathbb{Q}}$ cannot be larger than M.

S_3 – Integer Coefficients

Integrally: $M_1 = \langle \Lambda_3 \Gamma^\alpha \rangle \le \mathrm{Im}\, \alpha^3_{\mathbb{Z}} \le \langle \Lambda_3 \Gamma^\alpha \otimes \mathbb{Q} \rangle \cap \Lambda_3 \Gamma = M_2$

If $M_1 = M_2$: $S_3(\mathbb{Z}) = \mathrm{coker}\, \alpha^3_{\mathbb{Z}} = \Lambda_3 \Gamma / \langle \Lambda_3 \Gamma^\alpha \rangle$ is free.

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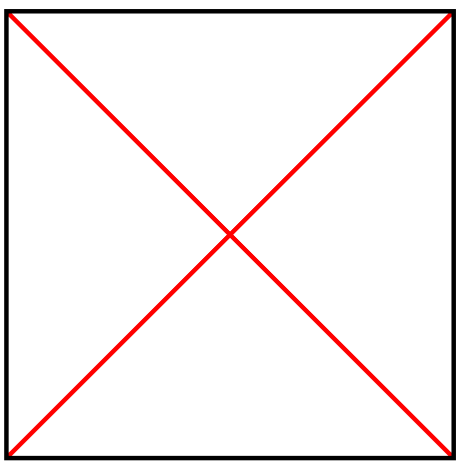
Otherwise: try sharper lower bound

$M_1' = \langle b_1^\theta \wedge b_2^\theta \wedge v | \theta \in I_1, \langle b_1^\theta, b_1^\theta \rangle = \Gamma^\theta, v \in \Gamma \rangle$.



3-chains can close on intersection 2-tori.

If $M_1' = M_2$:

$S_3(\mathbb{Z}) = \operatorname{coker} \alpha^3_{\mathbb{Z}} = \Lambda_3 \Gamma / M_2$ is free.

Does Torsion in $H_2(A)$ lift?

$$0 \longrightarrow S_3(\mathbb{Z}) \longrightarrow H^3(\Omega;\mathbb{Z}) \longrightarrow H_2(A;\mathbb{Z}) \xrightarrow{\alpha_{\mathbb{Z}}^4} H_2(\mathbb{T}^6;\mathbb{Z}) \longrightarrow 0$$

As $H_2(A;\mathbb{Z})$ may have torsion, the correct solution is not obvious.

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has two solutions, $X = \mathbb{Z}$ and $X = \mathbb{Z} \oplus \mathbb{Z}_2$.

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No rational arguments will help to distinguish these cases . . .

$\mathbf{S_3 - F_p}$ Coefficients

Suppose $S_3(\mathbb{Z})$ is free; Want to show: $rk(S_3(\mathbb{Q})) = rk(S_3(\mathbb{F}_p))$

We can write $A \longrightarrow \mathbb{T}^6$ as a composite:

$$A \xrightarrow{\Delta} A \times \ldots \times A \xrightarrow{a_1 \times \ldots \times a_6} S^1 \times \ldots \times S^1 \xrightarrow{i} \mathbb{T}^6$$

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In homology, we get

$$H_3(A) \xrightarrow{\Delta *} \bigoplus_{r_1+\cdots+r_6=3} H_{r_1}(A) \otimes \cdots \otimes H_{r_6}(A)$$
$$\xrightarrow{a_1 * \otimes \cdots \otimes a_6 *} \bigoplus_{r_1+\cdots+r_6=3} H_{r_1}(S^1) \otimes \cdots \otimes H_{r_6}(S^1) \xrightarrow{i*} H_3(\mathbb{T}^6)$$

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Since $H_k(S^1) = 0$ for $k \geq 2$, only $H_0(A)$ and $H_1(A)$ can contribute. For those dimensions, $H_k(A)$ is free; passing from $\mathbb{Q}$ to $\mathbb{F}_p$ cannot change the rank of $\operatorname{Im} \alpha^3$.

Torsion in $H_2(A)$ lifts!

$$0 \longrightarrow S_3(R) \longrightarrow H^3(\Omega; R) \longrightarrow H_2(A; R) \xrightarrow{\alpha_R^4} H_2(\mathbb{T}^6; R) \longrightarrow 0$$

Suppose $S_3(\mathbb{Z})$ is free, so that $rk(S_3(\mathbb{Q})) = rk(S_3(\mathbb{F}_p))$.

From the universal coefficient theorem, we have

$$rk(H^k(\Omega; \mathbb{F}_p)) - rk(H^k(\Omega; \mathbb{Q})) = T_p(H^k(\Omega; \mathbb{Z})) + T_p(H^{k+1}(\Omega; \mathbb{Z}))$$
$$rk(H_k(A; \mathbb{F}_p)) - rk(H_k(A; \mathbb{Q})) = T_p(H_k(A; \mathbb{Z})) + T_p(H_{k-1}(A; \mathbb{Z}))$$

where $T_p(X)$ is the rank of p-torsion in X.

Torsion in $H_2(A)$ lifts!

$$0 \longrightarrow S_3(R) \longrightarrow H^3(\Omega; R) \longrightarrow H_2(A; R) \xrightarrow{\alpha_R^4} H_2(\mathbb{T}^6; R) \longrightarrow 0$$

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where $T_p(X)$ is the rank of p-torsion in X. As a result, we get

$$rk(H^3(\Omega; \mathbb{F}_p)) - rk(H^3(\Omega; \mathbb{Q})) = T_p(H_2(A; \mathbb{Z}))$$
$$Torsion(H^3(\Omega; \mathbb{Z})) = Torsion(H_2(A; \mathbb{Z})).$$

Result

Consider a dimension 3, co-dimension 3 canonical projection tiling, and suppose $S_3(\mathbb{Z})$ is free. For any coefficient field $R = \mathbb{Q}$ or $\mathbb{F}_p$, we have

$$rk(H^k(\Omega; R)) = rk(H_{6-k-1}(A; \mathbb{Z})) + rk(S_k(R)) + rk(S_{k+1}(R)) - \binom{6}{k+1}.$$

For $R = \mathbb{Z}$, the same relation holds for the free ranks. Moreover,

$$Torsion(H^2(\Omega; \mathbb{Z})) = Torsion(S_2(\mathbb{Z}))$$

$$Torsion(H^3(\Omega; \mathbb{Z})) = Torsion(H_2(A; \mathbb{Z}))$$

All other $H^k(\Omega; \mathbb{Z}))$ are free.

Examples

Cohomology of some icosahedral tilings from the literature:

H^3	H^2	H^1	H^0	χ	planes	Γ	
$\mathbb{Z}^{20} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{16}$	$\mathbb{Z}^7$	$\mathbb{Z}$	10	5-fold	F	Danzer
$\mathbb{Z}^{181} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	120	mirror	P	Ammann-Kramer
$\mathbb{Z}^{331} \oplus \mathbb{Z}_2^{20} \oplus \mathbb{Z}_4$	$\mathbb{Z}^{102} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	240	mirror	F	dual can. D_6
$\mathbb{Z}^{205} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{72}$	$\mathbb{Z}^7$	$\mathbb{Z}$	145	3,5-fold	F	canonical D_6

Even the simplest of all icosahedral tilings have torsion!

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Formulae have to be evaluated by computer (GAP programs).

Combinatorics of intersection tori are determined with (descendants of) programs from the GAP package Cryst (B. Eick, F. Gähler, W. Nickel, Acta Cryst. A53, 467-474 (1997)).