The Meyer property on substitution point sets

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(Joint work with Boris Solomyak)

Outline

Definitions

- $\Lambda \subseteq \mathbb{R}^d$ is a Delone set if Λ is uniformly discrete & relatively dense
- Λ has finite local complexity(FLC) if for every $R > 0$, there are finitely many translational classes of clusters whose supports lie in a ball of radius R.
- Λ is a Meyer set if Λ is a Delone set and $\Lambda - \Lambda \subset \Lambda + F$, where F is a finite set (almost lattice).

Definitions

- A cut and project scheme is a collection of spaces and mappings for which

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\
 & & \bigcup & & \\
L & \longleftarrow & \tilde{L} & \longrightarrow & L^* \\
 & & & & \\
x & \longleftarrow & (x, x^*) & \longrightarrow & x^* ,
\end{array}
$$

  where
  (1) H : a locally compact Abelian group
  (2) $\tilde{L}$: a lattice in $\mathbb{R}^d \times H$
  (3) π_1 and π_2 are canonical projections such that $\pi_1|_{\tilde{L}}$ is 1-1 and $\pi_2(\tilde{L})$ is dense in H.

- $\Lambda(W) := \{\pi_1(x) \in \mathbb{R}^d | x \in \tilde{L}, \pi_2(x) \in W\}$, where $W \subset H$.

  $\Lambda(W)$ is a model set *if* $\overline{W}$ is compact and $W^\circ \neq \emptyset$.

  Γ is an inter model set *if* $\Lambda(W^\circ) \subset \Gamma \subset \Lambda(W)$, where W is compact, $W^\circ \neq \emptyset$, and $\overline{W^\circ} = W$.

Definitions

Model sets $\subset$ Meyer sets $\subset$ Delone sets with FLC $\subset$ Delone sets

Example

- (Non-Meyer sets):

$$\{n + \frac{1}{n} : n \in \mathbb{Z} \backslash \{0\}\}, \quad \{n + \frac{1}{6} \sin \frac{2\pi n}{\sqrt{2}} : n \in \mathbb{Z}\}$$

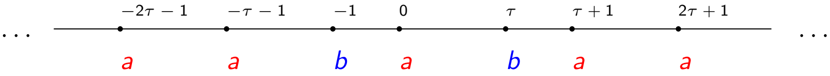
- (Meyer set):

$$\Lambda = \{a + b\tau \in \mathbb{Z}[\tau] : a - b\frac{1}{\tau} \in [0, 1]\}, \text{ where } \tau^2 - \tau - 1 = 0\,.$$

Example (Fibonacci substitution point set)

$$\begin{array}{ccc} & \Phi & \\ a & \longrightarrow & aba \\ b & \longrightarrow & ab \end{array}$$

$$\begin{array}{c} b \,|\, a \\ a\ b \,|\, a\ b\ a \\ \vdots \\ \cdots b\ a\ b\ a\ a\ b\ a\ a\ b \,|\, a\ b\ a\ a\ b\ a\ b\ a\ a \cdots \end{array}$$



where $\tau^2 - \tau - 1 = 0$.

$$\begin{array}{rcl} \Lambda_a & = & \tau^2\Lambda_a \ \cup\ (\tau^2\Lambda_a + \tau^2) \ \cup\ \tau^2\Lambda_b \\ \Lambda_b & = & (\tau^2\Lambda_a + \tau) \ \cup\ (\tau^2\Lambda_b + \tau) \end{array}$$

$\boldsymbol{\Lambda} := (\Lambda_a, \Lambda_b)$ is called a substitution Delone multi-colour set.

Substitution point sets

$\mathbf{\Lambda}$ is a substitution Delone multi-colour set
if a Delone multi-colour set $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ satisfies

$$\Lambda_i = \bigcup_{j \le m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m \tag{1}$$

with an expansive linear map Q and finite sets $\mathcal{D}_{ij}$.

$\mathbf{\Lambda}$ is primitive *if* the substitution matrix S, with $S_{ij} = \sharp(\mathcal{D}_{ij})$, is primitive, i.e. $S^l > 0$ for some $l > 0$.

$\mathbf{\Lambda}$ is representable for a tiling *if* $\mathcal{T} = \{x + T_i | x \in \Lambda_i, 1 \le i \le m\}$ is a tiling for some finite set of tiles $\{T_1, \cdots, T_m\}$.

Characterization of Meyer sets

Theorem (Meyer '72, Lagarias '95, Moody '97)

Let Λ be a Delone set. TFAE

1. Λ is a Meyer set.
2. $\Lambda - \Lambda$ is uniformly discrete.
3. Λ is a subset of a model set.

Spectral properties on Meyer sets

Theorem (Strungaru '05)

If Λ is a Meyer set (assuming the existence of autocorrelation), then the set of Bragg peaks is relatively dense.

Lagarias's questions '00:

Let Λ be a Delone set with FLC & repetitivity.

- If Λ is pure point diffractive, must Λ be a Meyer set?
- If Λ is pure point diffractive and a primitive representable substitution Delone set, must Λ be a Meyer set?

Spectral properties on Meyer sets

Theorem (Lee-Solomyak '08)

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ be a primitive representable substitution Delone multi-colour set in $\mathbb{R}^d$ of an expansive linear map ϕ with FLC. Then TFAE

1. The set of Bragg peaks for Λ_i is relatively dense.
2. The set of eigenvalues for $(X_{\mathbf{\Lambda}}, \mathbb{R}^d, \mu)$ is relatively dense.
3. $\bigcup_{i=1}^m \Lambda_i$ is a Meyer set.

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Corollary (Lee-Solomyak '08)

Let $\boldsymbol{\Lambda} = (\Lambda_i)_{i \le m}$ be a primitive representable substitution Delone multi-colour set in $\mathbb{R}^d$ of an expansive linear map ϕ with FLC. If $\boldsymbol{\Lambda}$ is pure point diffractive, then $\bigcup_{i=1}^m \Lambda_i$ is a Meyer set.

Deformed Meyer sets

Theorem (Lee-Moody '08)

Let $\Lambda \subset \mathbb{R}^d$ be a Meyer set and let $f : \langle \Lambda \rangle_{\mathbb{Z}} \to \mathbb{R}^d$ be any $\mathbb{Z}$-homomorphism. If $f(\Lambda)$ is untied, then $f(\Lambda)$ is a Meyer set in $\mathbb{R}^d$.

We say that $S \in \mathbb{R}^d$ is untied if there is no hyperplane (i.e. a $(d-1)$-dimensional subspace) H of $\mathbb{R}^d$ such that $S \subset H + B_r(0)$ for some $r > 0$.

Pisot number and Pisot family

Definition

- A Pisot number(Pisot-Vijayaraghavan number) θ is an algebraic integer $\theta > 1$ whose algebraic conjugates θ' are all less than 1 in absolute value.
- A Pisot family is a set of algebraic integers $\Theta = \{\theta_1, \cdots, \theta_n\}$ such that for each $1 \leq i \leq n$, every algebraic conjugate γ of θ_i with $|\gamma| \geq 1$ is in Θ.

We assume that

- $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$ is a primitive representable substitution Delone multi-colour set in $\mathbb{R}^d$ of an expansive linear map ϕ with FLC.
- $\Xi := \bigcup_{i \leq m} (\Lambda_i - \Lambda_i)$.

Meyer property on substitutions

Theorem (Kenyon '96)

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ be in $\mathbb{R}^2$ with an expansive factor $\theta (= \phi)$.

1. If $\theta \notin \mathbb{R}$, then $\Xi \subset \mathbb{Z}[\theta]\alpha$ for some $\alpha \in \mathbb{C}$.
2. If $\theta \in \mathbb{R}$, then $\Xi \subset \mathbb{Z}[\theta]\alpha_1 + \mathbb{Z}[\theta]\alpha_2$ for some basis $\{\alpha_1, \alpha_2\}$ of $\mathbb{R}^2$.

θ is a Pisot number if and only if $\cup_{i=1}^m \Lambda_i$ is a Meyer set.

Meyer property on substitutions

Theorem (Kenyon '94, Solomyak '06)

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$ be in $\mathbb{R}^d$ with an expansive factor $\theta (= \phi)$. Then

1. $\Xi \subset \mathbb{Z}[\theta]\alpha_1 + \cdots + \mathbb{Z}[\theta]\alpha_d$ for some basis $\{\alpha_1, \cdots, \alpha_d\}$ in $\mathbb{R}^d$.
2. θ is a Pisot number if and only if $\cup_{i=1}^{m} \Lambda_i$ is a Meyer set.

Meyer property on substitutions

Theorem (Környei '87)

If $\mathbf{\Lambda}$ in $\mathbb{R}^d$ with expansion map ϕ is a Meyer set, then the set of all the eigenvalues of ϕ forms a Pisot family.

Question: Does the Pisot family condition on ϕ force $\mathbf{\Lambda}$ to be a Meyer set?

Meyer property on substitutions

Theorem (Lee-Solomyak '08)

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ be in $\mathbb{R}^d$ with an expansive linear map ϕ. Suppose that ϕ is diagonalizable over $\mathbb{C}$ and all the eigenvalues of ϕ are distinct. Then

1. $\Xi \subset \mathbb{Z}[\phi]\alpha_1 + \cdots + \mathbb{Z}[\phi]\alpha_K$, where $\{\alpha_1, \cdots, \phi^{m_1 - 1}\alpha_1, \cdots, \alpha_K, \cdots, \phi^{m_K - 1}\alpha_K\}$ forms a basis of $\mathbb{R}^d$ for some $m_1, \cdots, m_K \in \mathbb{Z}_+$.
2. All the eigenvalues of ϕ form a Pisot family if and only if $\cup_{i=1}^m \Lambda_i$ is a Meyer set.

SKETCH OF PROOF.

Meyer property on substitutions

SKETCH OF PROOF.

[Case I] Assume that all eigenvalues of a diagonal matrix ϕ are algebraic conjugates of each other.

Consider a control point set $\mathcal{C}$ of $\mathbf{\Lambda}$ satisfying $\phi\mathcal{C} \subset \mathcal{C}$. Let $\widetilde{\mathcal{C}} := \langle \mathcal{C} \rangle_{\mathbb{Z}}$. By the structure theorem,

$$\widetilde{\mathcal{C}} \subset \mathbb{Q}[\phi]\gamma_1 \oplus \cdots \oplus \mathbb{Q}[\phi]\gamma_L$$

is a module over $\mathbb{Q}[\phi]$. Choose $\beta \in \mathcal{C}$ whose each entry is non-zero. Define $\pi : \widetilde{\mathcal{C}} \to \mathbb{Q}[\phi]\beta$ a module homomorphism. Let $\mathcal{C}_\infty := \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}$. Define

$$\pi' : \mathcal{C}_\infty \to \mathbb{Q}[\phi]\beta$$

such that $\pi'(x) = \phi^{-k}\pi\phi^k(x)$ for $x \in \phi^{-k}\mathcal{C}$.

Meyer property on substitutions

We show the following steps.

1. π' is uniformly continuous on $\mathcal{C}_\infty$ by showing Hölder's continuity.
2. Extend π' on $\mathbb{R}^d$.
3. $\pi'|_{x+E_{\lambda_{min}}}$ is affine linear.
4. π' is affine linear by using the algebraic conjugacy of all eigenvalues.
5. $\beta, \phi\beta, \cdots, \phi^{d-1}\beta$ are linearly independent in $\mathbb{R}^d$ over $\mathbb{R}$ Thus π' is an isomorphism.
6. $\mathcal{C} \subset \mathbb{Z}[\phi]\alpha$, where $\{\alpha, \phi\alpha, \cdots, \phi^{d-1}\alpha\}$ is a basis of $\mathbb{R}^d$, for some $\alpha \in \mathbb{R}^d$, by FLC and the isomorphism of π'.

[Case II] Assume that all eigenvalues of ϕ are distinct.

We decompose ϕ into block matrices in such a way that eigenvalues of each block matrix are all the eigenvalues of ϕ which are algebraic conjugates of an eigenvalue of ϕ. We use the similar method as **Case I**.

$\square$

THANK YOU!