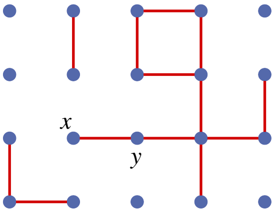
Bielefeld, 4 July 2008

Random colourings of quasiperiodic graphs: Ergodic and spectral properties

see also arXiv:0709.0821

Peter Müller, Christoph Richard

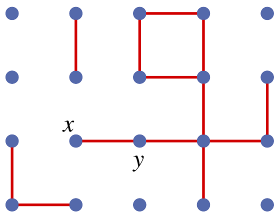
1 Motivation: Laplacian on percolation graphs in $\mathbb{Z}^d$



random graph with realisations
$G_\omega = (\mathbb{Z}^d, \mathcal{E}_\omega)$

- $\{x, y\} \in \mathcal{E}_\omega$ with probability $p \in [0,1]$, if $|x - y| = 1$, and zero otherwise
- different edges independently and identically distributed

1 Motivation: Laplacian on percolation graphs in $\mathbb{Z}^d$



random graph with realisations
$G_\omega = (\mathbb{Z}^d, \mathcal{E}_\omega)$

- $\{x,y\} \in \mathcal{E}_\omega$ with probability $p \in [0,1]$, if $|x-y| = 1$, and zero otherwise
- different edges independently and identically distributed

Laplacian:

$$(\Delta_\omega \varphi)(x) := \sum_{y:\{x,y\}\in\mathcal{E}_\omega} [\varphi(x) - \varphi(y)]$$

$\forall x \in \mathbb{Z}^d \quad \forall \varphi \in \ell^2(\mathbb{Z}^d)$

- a.s. self-adjoint, ergodic w.r.t. $\mathbb{Z}^d$-translations
- $\operatorname{spec} \Delta_\omega = [0, 4d]$ a.s.
- integrated density of states:

$$N(E) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\#\text{ eigenvalues of } \Delta_{\omega,\Lambda} \text{ not exceeding } E}{|\Lambda|}$$

$$= \int \mathbb{P}(d\omega') \, \langle \delta_0, \chi_{]-\infty,E]}(\Delta_{\omega'})\delta_0 \rangle$$

"number density of eigenvalues not exceeding E"

Theorem

(a) [Kirsch / Müller 06]
If $p < p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

(Lifshits tail)

(b) [Müller / Stollmann 07]
If $p > p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln[N(E) - N(0)]}{\ln E} = \frac{d}{2}$$

(van Hove asymptotics)

Theorem

(a) [Kirsch / Müller 06]
If $p < p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln|\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

(Lifshits tail)

(b) [Müller / Stollmann 07]
If $p > p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln[N(E) - N(0)]}{\ln E} = \frac{d}{2}$$

(van Hove asymptotics)

Idea of proof for (a):

linear clusters dominate!

Upper bound:

smallest positive eigenvalue for cluster containing the origin ↓

$$N(E) - N(0) \leq \mathbb{P}\{E_1(\mathcal{C}_\omega) \leq E\}$$

$$\leq \mathbb{P}\{|\mathcal{C}_\omega|^{-2} \leq E\}$$

Cheeger inequality: ↑
$E_1(\mathcal{C}_\omega) \geq |\mathcal{C}_\omega|^{-2}$

$$\leq \exp\{-\text{const.}\, E^{-1/2}\}$$

↗
exponential decay of cluster-size distribution for $p < p_c$

Lower bound:
Only keep contributions from linear clusters in $\mathbb{P}$

Theorem

(a) [Kirsch / Müller 06]
If $p < p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

(Lifshits tail)

(b) [Müller / Stollmann 07]
If $p > p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln[N(E) - N(0)]}{\ln E} = \frac{d}{2}$$

(van Hove asymptotics)

Generalisation of (a) to Cayley graphs: [Antunović / Veselić 07]

Idea of proof for (a):

linear clusters dominate!

Upper bound:

smallest positive eigenvalue for cluster containing the origin ↓

$$N(E) - N(0) \le \mathbb{P}\{E_1(\mathcal{C}_\omega) \le E\}$$

$$\le \mathbb{P}\{|\mathcal{C}_\omega|^{-2} \le E\}$$

Cheeger inequality: $E_1(\mathcal{C}_\omega) \ge |\mathcal{C}_\omega|^{-2}$ ↑

$$\le \exp\{-\text{const.}\ E^{-1/2}\}$$

↗ exponential decay of cluster-size distribution for $p < p_c$

Lower bound:

Only keep contributions from linear clusters in $\mathbb{P}$

Theorem

(a) [Kirsch / Müller 06]
If $p < p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

(Lifshits tail)

(b) [Müller / Stollmann 07]
If $p > p_c$ then

$$\lim_{E \downarrow 0} \frac{\ln[N(E) - N(0)]}{\ln E} = \frac{d}{2}$$

(van Hove asymptotics)

Generalisation of (a) to Cayley graphs: [Antunović / Veselić 07]

Idea of proof for (a):

linear clusters dominate!

Upper bound:

smallest positive eigenvalue for cluster containing the origin ↓

$$N(E) - N(0) \leq \mathbb{P}\{E_1(\mathcal{C}_\omega) \leq E\}$$

$$\leq \mathbb{P}\{|\mathcal{C}_\omega|^{-2} \leq E\}$$

Cheeger inequality: $E_1(\mathcal{C}_\omega) \geq |\mathcal{C}_\omega|^{-2}$ ↑

$$\leq \exp\{-\text{const. } E^{-1/2}\}$$

↗ exponential decay of cluster-size distribution for $p < p_c$

Lower bound:
Only keep contributions from linear clusters in $\mathbb{P}$

This talk: Replace underlying lattice $\mathbb{Z}^d$ by aperiodic graph

2 Graphs and ergodic dynamical systems

Simple, infinite graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ with uniformly discrete vertex set (UDV)

Two graph operations:

- translation by $x \in \mathbb{R}^d$: $x + G$
- truncation to compact $B \subset \mathbb{R}^d$: $G \wedge B$ (pattern)

G has finite local complexity (FLC):

$\forall \rho > 0$ fixed, G has only finitely many different (up to translations) vertex-centred ball patterns of radius ρ.

2 Graphs and ergodic dynamical systems

Simple, infinite graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ with uniformly discrete vertex set (UDV)

Two graph operations:

- translation by $x \in \mathbb{R}^d$: $x + G$
- truncation to compact $B \subset \mathbb{R}^d$: $G \wedge B$ (pattern)

G has finite local complexity (FLC):
$\forall \rho > 0$ fixed, G has only finitely many different (up to translations) vertex-centred ball patterns of radius ρ.

Distance between G and G':

smallest $\varepsilon > 0$ such that $G \wedge B_{1/\varepsilon}$ and $G' \wedge B_{1/\varepsilon}$ differ by a translation of length at most ε.

(B_r: ball in $\mathbb{R}^d$ of radius r around origin)

Hull of G_0: $X := \overline{\{x + G : x \in \mathbb{R}^d\}}$ (complete metric space, if G_0 has UDV)

$\Rightarrow$ topological dynamical system $(X, \mathbb{R}^d, +)$

Lemma

X compact $\iff$ G has FLC

Theorem (Radin/Wolff, Schlottmann, Lee/Moody/Solomyak)

Let G_0 be a graph with UDV and FLC. Fix an ergodic probability measure μ on X. Then we have for all $\varphi \in L^1(X,\mu)$

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} dx\, \varphi(x + \tilde{G}) = \int_X d\mu(G)\, \varphi(G)$$

for μ-a.a. $\tilde{G} \in X$. [If X uniquely ergodic and φ continuous, then this holds $\forall \tilde{G} \in X$.]

Corollary

Let P be a pattern of some $G \in X$. Then the pattern frequency

$$\nu(P) := \lim_{n\to\infty} \frac{\#\text{ occurrences of } P \text{ in } \tilde{G} \wedge B_n}{\mathrm{vol}(B_n)}$$

exists for μ-a.a. $\tilde{G} \in X$. [X uniquely ergodic: $\forall \tilde{G} \in X$.]

③ Randomly coloured graphs

Here only edge colouring!

$\mathbb{A}$: finite set of colours

$\mathbb{P}_0$: probab. measure on $\mathbb{A}$

Colourings for a given graph G:

$\Omega_G := \times_{e \in \mathcal{E}_G} \mathbb{A}$, $\quad \omega = (\omega_e)_{e \in \mathcal{E}_G}$

with i.i.d. colours $\mathbb{P} := \bigotimes_{e \in \mathcal{E}_G} \mathbb{P}_0$

Space of couloured graphs: $\hat{X} := \{G_\omega := (G, \omega) : G \in X, \omega \in \Omega_G\}$

Theorem (Ergodic theorem)

Assume G_0 obeys UDV and FLC. Fix an ergodic probab. measure μ on X. Then $\exists_1$ ergodic probab. measure $\hat{\mu}$ on $\hat{X}$ such that for all $\varphi \in L^1(\hat{X}, \hat{\mu})$

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} dx\, \varphi(x + \tilde{G}_{\tilde{\omega}}) = \int_{\hat{X}} d\hat{\mu}(G_\omega)\, \varphi(G_\omega) = \int_X d\mu(G) \int_{\Omega_G} d\mathbb{P}_G(\omega)\, \varphi(G_\omega)$$

for $\hat{\mu}$-a.a. $\tilde{G}_{\tilde{\omega}} \in \hat{X}$. [If X uniquely ergodic and φ continuous, then this holds $\forall \tilde{G} \in X$ and $\mathbb{P}_G$-a.a. $\tilde{\omega} \in \Omega_{\tilde{G}}$.]

- generalisation of [Hof 98]
- proof uses ergodic theorem on X and strong law of large numbers

4 Finite-range operators on randomly coloured graphs

Definition

Let G_ω be a coloured graph, H_{G_ω} be bounded and self-adjoint on $\ell^2(\mathcal{V}_G)$.

H_{G_ω} is covariant of finite range $R > 0$ $\quad :\Longleftrightarrow$

1. $\langle \delta_v, H_{G_\omega} \delta_w \rangle = 0, \quad$ if $|v - w| \geq R$
2. $\langle \delta_v, H_{G_\omega} \delta_w \rangle = 0 = \langle \delta_{x+v}, H_{G_\omega} \delta_{x+w} \rangle = 0, \quad$ if for both $u = v$ and $u = w$:

$$x + \Big(G_\omega \wedge B_R(u) \Big) = \Big(G_\omega \wedge B_R(x+u) \Big)$$

Matrix elements depend only on local pattern!

Existence of the integrated density of states N, selfaveraging and non-randomness of the spectrum

Theorem

Fix ergodic prob. measure $\hat{\mu}$ on $\hat{X}$. Assume there is $R > 0$ s.t. $\forall G_\omega \in \hat{X}$ H_ω is covariant of finite range R. Then

(1) $\exists N : \mathbb{R} \to \mathbb{R}_{\geq 0}$, non-decreasing, s.t. for $\hat{\mu}$-a.e. $G_\omega \in \hat{X}$

$$N(E) = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{v \in \mathcal{V}_G \cap B_n} \langle \delta_v, \chi_{]-\infty,E]}(H_{G_\omega})\delta_v \rangle \qquad (*)$$

for all continuity points E of N. [Uniquely ergodic: $\forall G \in X$ and $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$]

(2) for $\hat{\mu}$-a.e. $G_\omega \in \hat{X}$: $\mathrm{spec}(H_{G_\omega}) = \mathrm{spec}_{ess}(H_{G_\omega}) = \mathrm{supp}(dN)$

[Uniquely ergodic and $\nu(P) > 0$ for all patterns P in G, $G \in X$: $\forall G \in X$ and $\mathbb{P}_G$-a.a. $\omega \in \Omega_G$]

Without colouring: (1) [Hof95, Lenz/Stollmann 05, Lenz/Peyerimhoff/Veselić 07]
(2) [Lenz/Stollmann 03]; Uniform convergence in E in $(*)$: [Lenz/Veselić]

5 Lifshits tails for the Laplacian

Assumptions:

1. Graph G_0 with
   - FLC, UDV
   - connected with vertex density >0
   - maximum bond length: $\ell_{max} := \sup\{|u-v| : u, v \in \mathcal{E}_{G_0}\} < \infty$
2. $\mathbb{A} = \{0,1\}$, $\mathbb{P}_0 = \text{Bernoulli}(p)$ (bond probability $p \in]0,1[$)
3. $H_{G_\omega} = \Delta_\omega$ Laplacian for $G_\omega \in \hat{X}$

   $$(\Delta_\omega \varphi)(v) := \sum_{u \in \mathcal{V}_G :\, \omega_{\{u,v\}}=1} [\varphi(v) - \varphi(u)]$$

   self-adjoint, bounded, covariant of finite range
4. $N(E)$: int. dens. of states w.r.t. ergodic prob. measure $\hat{\mu}$ on $\hat{X}$

5 Lifshits tails for the Laplacian

Assumptions:

1. Graph G_0 with
   - FLC, UDV
   - connected with vertex density >0
   - maximum bond length: $\ell_{max} := \sup\{|u - v| : u, v \in \mathcal{E}_{G_0}\} < \infty$
2. $\mathbb{A} = \{0, 1\}$, $\mathbb{P}_0 = \text{Bernoulli}(p)$ (bond probability $p \in]0, 1[$)
3. $H_{G_\omega} = \Delta_\omega$ Laplacian for $G_\omega \in \hat{X}$

$$(\Delta_\omega \varphi)(v) := \sum_{u \in \mathcal{V}_G :\, \omega_{\{u,v\}} = 1} [\varphi(v) - \varphi(u)]$$

   self-adjoint, bounded, covariant of finite range
4. $N(E)$: int. dens. of states w.r.t. ergodic prob. measure $\hat{\mu}$ on $\hat{X}$

Theorem

Assume (1) to (4). Let $p \in]0, p_c[$ s.t. with uniform constants D_p, λ_p

$$\mathbb{P}_G(|C_\omega^{(v)}| \geq n) \leq D_p \, e^{-\lambda_p n} \qquad (*)$$

$\forall n \in \mathbb{N}$ (exponential decay of cluster-size distribution). Then

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -\frac{1}{2}$$

Lemma

Let d_{max} be the maximal vertex degree of G_0 and assume $0 \leq p < \frac{1}{d_{max} - 1}$. Then $(*)$ holds.

6 Outlook

- prove exponential decay of cluster-size distribution
  for all $p \in]0, p_c[$

  Meshikov Theorem? Aizenman / Barsky result?

- FLC-condition versus Delone sets

- generalisation to random tilings (OK!)

- spectral asymptotics of N in the percolating phase?

  $\rightarrow$ further conditions needed on G_0 ?!

- Dirichlet Laplacians?