The "Local-Global Principle" for Model Sets

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Workshop "Aspects of Aperiodic Order"

Substitutions I

- Sequences: finite alphabet & substitution rule

e.g., $\{a, b\}$ $\quad \sigma_1 : a \mapsto aaabbbbbbbb = a^3b^8,\ b \mapsto abbb$

$$\sigma_2 : a \mapsto aaabbbb = a^3b^4,\ b \mapsto aabbb$$

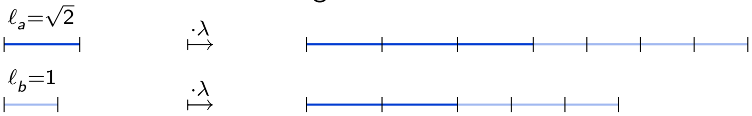
$$b.a \overset{\sigma_1}{\mapsto} abbb.aaabbbbbbbb \overset{\sigma_1}{\mapsto} \ldots bb.aa \ldots$$

- Inflation factor λ is PF-eigenvalue of substitution matrix

$$\begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \quad \text{for both} \quad \lambda = 3 + 2\sqrt{2} \approx 5.828$$

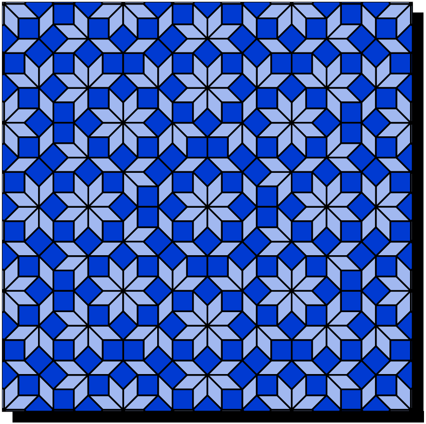
- sub. matrix primitive: components of (left) PF-eigenvector positive

$\Rightarrow$ lengths of intervals



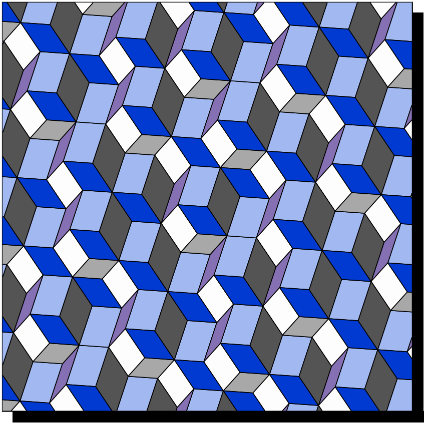
Substitutions I

- 2-dimensional tilings, e.g., Ammann-Beenker ($\lambda = 1 + \sqrt{2}$)



Substitutions I

- ...or Conch and its dual partner Nautilus



(6 prototiles, inflation factor not PV-number,
namely $\approx -0.727 - i\,0.934$ resp. $\approx 1.019 - i\,0.603$)

Substitutions II

Let $\varLambda_i$ be the set of left endpoints of intervals of type i

(... controlpoints of tiles of type i)

- The sets $\varLambda_i$ are contained (in a translate of)

$$\mathcal{L} = \langle (\bigcup_i \varLambda_i) - (\bigcup_i \varLambda_i) \rangle_{\mathbb{Z}}$$

- Also define $\mathcal{L}' = \langle \bigcup_i (\varLambda_i - \varLambda_i) \rangle_{\mathbb{Z}}$. Then $\mathcal{L}' \subset \mathcal{L}$.
- For sequences: $\mathcal{L}' \subset \mathcal{L} = \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}} \subset \mathbb{Q}(\lambda)$.

  Penrose (vert. = ctrlpts.): $\mathcal{L}' = 5\mathbb{Z}[\xi_5] \subset \mathcal{L} = \mathbb{Z}[\xi_5] \subset \mathbb{Q}(\xi_5)$.

  Ammann-Beenker (vert. = ctrlpts.): $\mathcal{L}' = \mathcal{L} = \mathbb{Z}[\xi_8] \subset \mathbb{Q}(\xi_8)$.

  In the last two examples, $\mathbb{Q}(\lambda)$ is maximal real subfield.

Substitutions II

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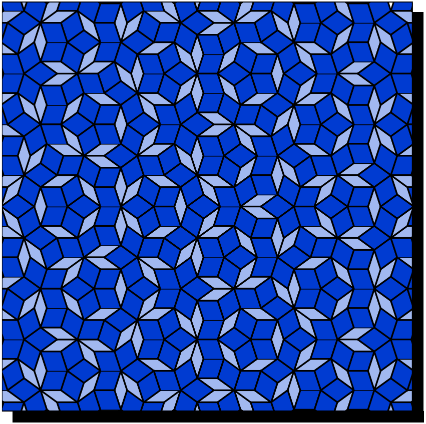
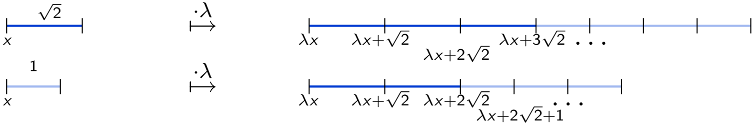


Substitutions II

Let Λ_i be the set of left endpoints of intervals of type i

(…controlpoints of tiles of type i)

- We assume that λ, $\mathcal{L}'$, $\mathcal{L}$ are contained in some algebraic number field K.
- Substitution operates on this field K.



- We write $\Lambda_i = \bigcup_j \lambda\,\Lambda_j + T_{ij}$ resp. $\underline{\Lambda} = \mathbf{\Theta}(\underline{\Lambda})$
  and $A_i = \bigcup_j \frac{1}{\lambda}\,A_j + \frac{1}{\lambda}\,T_{ji}$ resp. $\underline{A} = \mathbf{\Theta}^{\#}(\underline{A})$.

$\mathbf{\Theta}$ and its dual $\mathbf{\Theta}^{\#}$ operate on the (global) field K.

Local fields

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To learn something about the global field, we have to study (all) local fields!

- Local fields are the (nontrivial) completions of an algebraic number field (non-isomorphic).
- A Galois automorphism σ yields an Archimedean absolute value of $x \in K$ via $|\sigma(x)| \leadsto \mathbb{R}$ or $\mathbb{C}$

  K degree n, then r real and s complex non-isomorphic fields ($n = r + 2s$)
- Each prime ideal $\mathfrak{p}$ of the ring $\mathfrak{o}_K$ of integers of K an non-Archimedean absolue value $\leadsto$ $\mathfrak{p}$-adic number field $\mathbb{Q}_\mathfrak{p}$.

Prime ideals of $\mathfrak{o}_{\mathbb{Q}(\sqrt{2})} = \mathbb{Z}[\sqrt{2}]$: $(2) = (\sqrt{2})^2$ ramifies, $p \equiv \pm 1 \bmod 8$ split (prime ideals $(a \pm \sqrt{2})$ where $a^2 \equiv 2 \bmod p$), $p \equiv \pm 3 \bmod 8$ inert.

$\mathfrak{p}$-adic fields

Every prime ideal $\mathfrak{p}$ of the integers $\mathfrak{o}_K$ of an alg. number field K yields a (compl.) $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$ (w.r.t. ultrametric abs. val. $\|\cdot\|_\mathfrak{p}$).

(for $x \in K$: If $(x) = \prod_{\mathfrak{p} \in \mathbb{P}_K} \mathfrak{p}^{v_\mathfrak{p}}$, then $\|x\|_\mathfrak{p} = p^{-f \cdot v_\mathfrak{p}}$ with $p \in \mathfrak{p}$, f residue deg.)

Given $\mathbb{Q}_\mathfrak{p}$, the ring $\hat{\mathfrak{o}}_\mathfrak{p} = \{x \in \mathbb{Q}_\mathfrak{p} \mid \|x\|_\mathfrak{p} \leq 1\}$ is a discrete valuation ring with prime ideal $\mathfrak{m} = \{x \in \mathbb{Q}_\mathfrak{p} \mid \|x\|_\mathfrak{p} < 1\}$.

Every element π s.t. $\mathfrak{m} = \pi \hat{\mathfrak{o}}_\mathfrak{p}$ is called a uniformizer.

Every element $x \in \mathbb{Q}_\mathfrak{p}$ can (uniquely) be written as

$$x = \sum_{k=m}^{\infty} d_k \, \pi^k \qquad \text{with } d_k \in D \text{ and } m \in \mathbb{Z},$$

where D is a system of representatives of the residue field $\hat{\mathfrak{o}}_\mathfrak{p}/\mathfrak{m}$.

Contraction & Expansion

Depending on the local field:

Inflation factor λ is contraction ($\|\lambda\| < 1$), expansion ($\|\lambda\| > 1$) or $\|\lambda\| = 1$.

Then: $\frac{1}{\lambda}$ is expansion, contraction or $\|\frac{1}{\lambda}\| = 1$.

In a $\mathfrak{p}$-adic space $\mathbb{Q}_\mathfrak{p}$: If $\|\lambda\|_\mathfrak{p} = 1$, then λ is a unit.

Substitution by Salem number:

Consider the substitution $a \mapsto ab$, $b \mapsto ac$, $c \mapsto d$ and $d \mapsto b$. Then, the inflation factor $\lambda \approx 1.722$ is a Salem number, algebraic conjugates are ≈ 0.581 and $\approx -0.651 \pm i\,0.759$ (abs. value 1).

One can show: In the complex local field (arising from $\approx -0.651 \pm i\,0.759$, not a root of unity), the Λ_i are not bounded!

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Contraction & Expansion

Let

- G be product of "expansive" local fields,
- H be product of "contractive" local fields.

Then

- $\boldsymbol{\Theta}$ is expansion on G, contraction on H,
- $\boldsymbol{\Theta}^{\#}$ is expansion on H, contraction on G.

Expansion = proper substitution,

contraction $\rightsquigarrow$ IFS

$\Rightarrow$ unique nonempty compact solution of

$\boldsymbol{\Theta}$ on H (denoted by $\underline{\Omega}$) and

$\boldsymbol{\Theta}^{\#}$ on G (denoted by $\underline{A}$ – self-affine/similar prototiles).

CPS

Let

- G be product of all "expansive" local fields,
- H be product of all "contractive" local fields,
- diagonal embedding of $\mathcal{L}'_{\text{ext}} = \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m}\mathcal{L}'$ is a lattice in $G \times H$.

$\mathcal{L}$ vs. $\mathcal{L}'$ (resp. $\mathcal{L}_{\text{ext}}$ vs. $\mathcal{L}'_{\text{ext}}$):

choose control points s.t. $\mathcal{L} = \mathcal{L}'$ (resp. $\mathcal{L}_{\text{ext}} = \mathcal{L}'_{\text{ext}}$),

otherwise $H \; \rightarrow \; H \times \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$ (e.g., Penrose: $H = \mathbb{C} \times \mathrm{C}_5$).

Factor $\frac{1}{\lambda}$ needed to "fill" all of $\mathfrak{p}$-adic components in $G \times H$

(otherwise, only lattice in compactly generated subspace).

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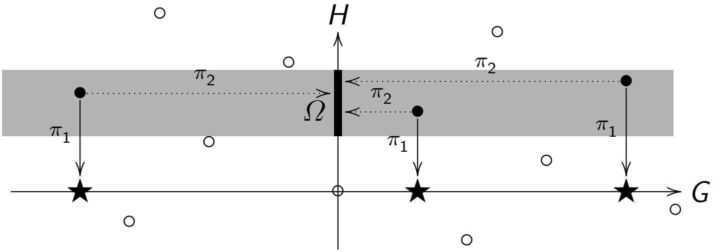

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For model sets:

all "contractive" & "expansive" local fields needed for density formula to hold.

Let $\underline{A} = \mathbf{\Theta}^{\#}(\underline{A})$ (attractor in G) and $\underline{\Omega} = \mathbf{\Theta}^{\star}(\underline{\Omega})$ (attractor in H), then for model set we must have

$$\sum_i \mu_G(A_i) \cdot \mu_H(\Omega_i) / \mu_{G\times H}\left(\text{FD}(\tilde{\mathcal{L}}'_{\text{ext}})\right) = 1.$$

(compare with *product formula* for absolute values)

⇒ "non-Pisot" substitutions never yield model sets.

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(compare with *product formula* for absolute values)

$\Rightarrow$ "non-Pisot" substitutions never yield model sets.

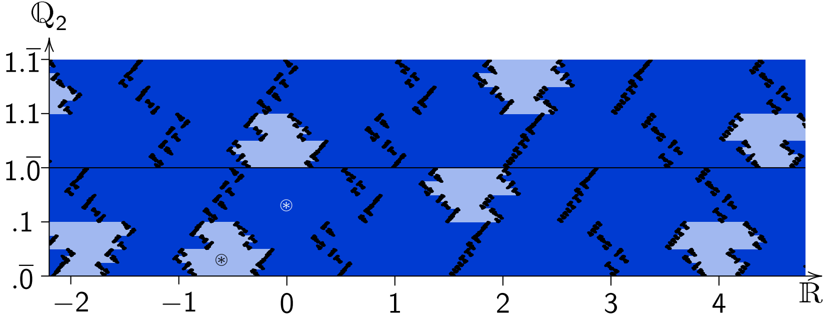
Dual Tilings

Let $\underline{\Omega}$ and $\underline{A}$ be as before, and define $\underline{\Upsilon} = \Lambda(\underline{A})$ in H

(repet. solution of expansive $\mathbf{\Theta}^{\#}$). Then:

$\underline{\Lambda}$ is a model set ($\underline{\Lambda} = \Lambda(\underline{\Omega})$) in G iff $\underline{\Upsilon} + \underline{\Omega}$ is a tiling of H.

"Internal" tiling for substitution $a \overset{\sigma}{\mapsto} aaba$, $b \overset{\sigma}{\mapsto} aa$:



Lattice revisited

In $\mathfrak{p}$-adic components where λ (resp. $\frac{1}{\lambda}$) is unit:

$\mathrm{cl}_{\mathbb{Q}_\mathfrak{p}} \mathcal{L}'$ (resp. $\mathrm{cl}_{\mathbb{Q}_\mathfrak{p}} \mathcal{L}'_{\mathrm{ext}}$, resp. $\mathrm{cl}_{\mathbb{Q}_\mathfrak{p}} \Lambda$) is compact clopen set.

e.g., $\sigma_1 : a \mapsto aaabbbbbbbb = a^3 b^8,\ b \mapsto abbb$

with $\ell_a = 2\sqrt{2}$ and $\ell_b = 1$

$\sigma_2 : a \mapsto aaabbbb = a^3 b^4,\ b \mapsto aabbb$

with $\ell_a = \sqrt{2}$ and $\ell_b = 1$

$\mathcal{L}'_1 = \langle 1, 2\sqrt{2} \rangle_{\mathbb{Z}}, \quad \mathcal{L}'_2 = \mathbb{Z}[\sqrt{2}]$

In both cases: $\mathrm{cl}_{\mathbb{Q}_\mathfrak{p}} \mathcal{L}'_{1,2} = \hat{\mathfrak{o}}_\mathfrak{p}$ for all $\mathfrak{p}$ with $2 \notin \mathfrak{p}$

$\mathrm{cl}_{\mathbb{Q}_{(\sqrt{2})}} \mathcal{L}'_2 = \hat{\mathfrak{o}}_{(\sqrt{2})} = (\sqrt{2})^2 \cup (\sqrt{2})^2 + \sqrt{2} \cup (\sqrt{2})^2 + 1 \cup (\sqrt{2})^2 + \sqrt{2} + 1,$

but $\mathrm{cl}_{\mathbb{Q}_{(\sqrt{2})}} \mathcal{L}'_1 = (\sqrt{2})^2 \cup (\sqrt{2})^2 + 1.$

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Adele Ring

Given an algebraic number field K, we may form the restricted product of all its (non-isomorphic) local fields $\rightsquigarrow$ ring of adeles $\mathbb{A}_K$:

$$\mathbb{A}_K = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\mathfrak{p}\in\mathbb{P}_K}{}' \ \mathbb{Q}_{\mathfrak{p}}$$

- (Diagonal embedding of) K is a lattice in $\mathbb{A}_K$.
- *Strong approximation theorem*:

  If we choose any local field, then K is dense in the (restricted) product with this local field removed.

Adelic CPS

$$\begin{array}{ccccc} G \times H & \longleftarrow & \mathbb{A}_K & \longrightarrow & H_{\mathbb{A}_K} = \prod'_{\mathbb{Q}_\mathfrak{p} \text{ not part of } G\times H} \mathbb{Q}_\mathfrak{p} \\ \text{dense } \cup & & \cup & & \cup \text{ dense} \\ K & \overset{\text{bijective}}{\longleftrightarrow} & K & \overset{\text{bijective}}{\longleftrightarrow} & K \end{array}$$

Lattice $\mathcal{L}'_{\text{ext}}$ in "original" CPS $(G, H, \mathcal{L}'_{\text{ext}})$ is model set with window $\text{cl}\,\mathcal{L}'$ (resp. $\text{cl}\,\mathcal{L}'_{\text{ext}}$) in the adelic CPS.

- We cannot use the same CPS for σ_1 and σ_2.
- Salem-substitution are "non-Pisot".

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Reciprocal Lattice

For diffraction, the *annihilator* $(\tilde{\mathcal{L}}'_{\text{ext}})^{\perp}$ yields location of Bragg peaks (group of eigenvalue of pp spectrum).

For algebraic number fields, it can be calculated via the *codifferent* $(\mathcal{L}')^{\wedge}$ of $\mathcal{L}'$:

$$(\tilde{\mathcal{L}}'_{\text{ext}})^{\perp} = \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} (\mathcal{L}')^{\wedge}.$$

Reciprocal Lattice

For diffraction, the *annihilator* $(\tilde{\mathcal{L}}'_{\text{ext}})^{\perp}$ yields location of Bragg peaks (group of eigenvalue of pp spectrum).

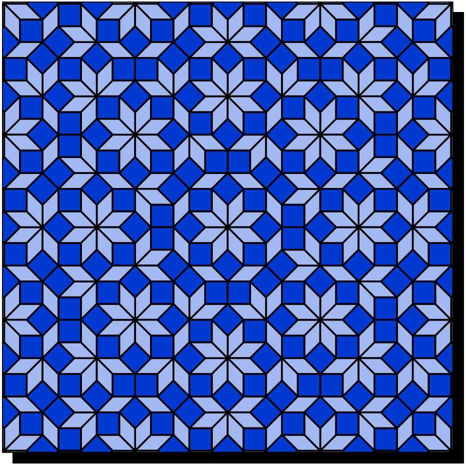
- $\mathbb{A}_K^* \cong \mathbb{A}_K$, $K^{\perp} \cong K$.
- In $\mathfrak{p}$-adic space: Fourier-transform of a characteristic function of a ball vanishes outside ball.
- Therefore, in the adelic CPS, the FT of the char. funct. of the window describing $\tilde{\mathcal{L}}'_{\text{ext}}$ is nonzero on a product of union of balls $\rightsquigarrow$ window for reciprocal lattice (w.r.t. adelic CPS).

e.g., $(\mathcal{L}'_2)^{\wedge} = \langle \frac{1}{2}, \frac{1}{2\sqrt{2}} \rangle_{\mathbb{Z}}$, note: $\hat{\mathfrak{o}}_{(\sqrt{2})} \overset{\text{FT}}{\rightsquigarrow} (\sqrt{2})^{-3}$

For $\mathcal{L}'_1$ with $\text{cl}_{\mathbb{Q}_{(\sqrt{2})}} \mathcal{L}'_1 = (\sqrt{2})^2 \cup (\sqrt{2})^2 + 1 \overset{\text{FT}}{\rightsquigarrow} (\sqrt{2})^{-3} \cup (\sqrt{2})^{-3} + \frac{1}{\sqrt{2}^5}$

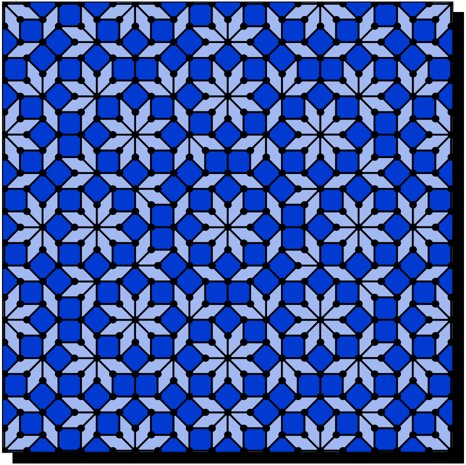
Visibility

The prototiles of the Ammann-Beenker tiling are a square and a rhomb (of sidelength 1).



Visibility

The vertices of the Ammann-Beenker tiling form a relatively dense and uniformly discrete point set.



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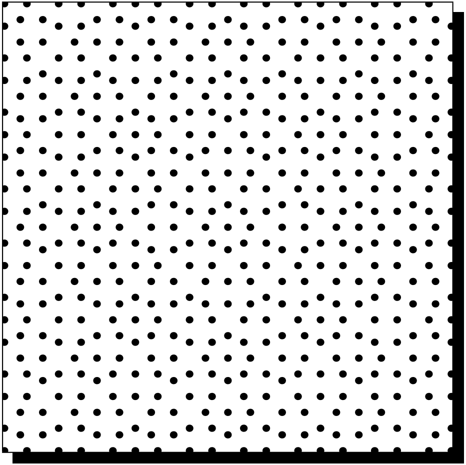
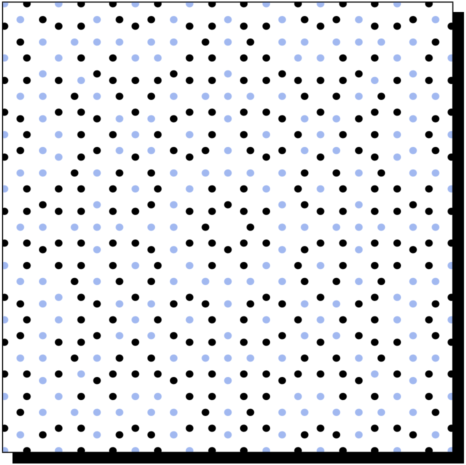


Visibility

One might ask: Which points are visible here?



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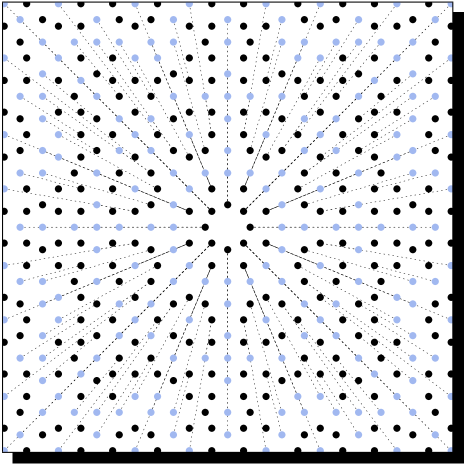
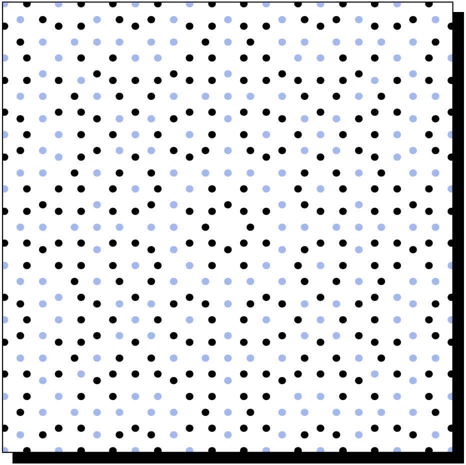


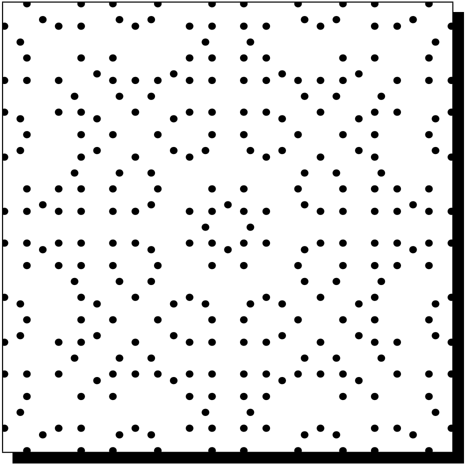
Visibility

One might ask: Which points are visible here? And what is their density?



Visibility

One might ask: Which points are visible here? And what is their density? How do they diffract?



Visibility

Density: $1/\zeta_{\mathbb{Q}(\sqrt{2})}(2) \approx 0.6969$ (about 57.73% of all AB-points)

Diffraction pure point (but structure has arbitrarily big holes).