14. Appendix. Cotorsion pairs. (Added October 2002).

It seems to be worthwhile to provide a reformulation of parts of the paper in terms of cotorsion pairs. These reformulations are based on notes [T2] of Jan Trlifaj written for his CRM lectures at Barcelona, October 2002, as well as conversations of the second author with him and with Lydia Angeleri-Hügel which were greatly appreciated (see also Trlifaj's Cortona Notes [T1] and the book [EJ] by Enochs and Jenda). Cotorsion pairs were introduced (for $R = \mathbb{Z}$) by Salce [S] in 1979, the corresponding paper is contained in the same volume as [R1]. The paper [R1] has as one of its main objectives the study of the pair $(\mathcal{C}, \mathcal{D})$ obtained from the tubular family of a tame hereditary algebra, and as we have seen in the present paper, a pair $(\mathcal{C}, \mathcal{D})$ with similar properties is obtained from any sincere stable separating tubular family. These pairs $(\mathcal{C}, \mathcal{D})$ are cotorsion pairs and there do exist by now several general results on cotorsion pairs which explain very well in which way properties of $(\mathcal{C}, \mathcal{D})$ and of $\omega = \mathcal{C} \cap \mathcal{D}$ are interrelated. For example, our proof of Theorem 5 uses left ω-approximations in order to obtain right ω-approximations, but this argument is a general feature of complete cotorsion pairs and has to be attributed to Salce [S].

The aim of this appendix is to recall relevant definitions and results and to outline properties of the pair $(\mathcal{C}, \mathcal{D})$ which are related by general observations concerning cotorsion pairs. Using the general theory, one should be able to rewrite (and may be even squeeze) some of the considerations of the paper. On the other hand, we hope that the following reformulations may help to see that the pairs $(\mathcal{C}, \mathcal{D})$ can serve as illuminating examples of cotorsion pairs.

Given a class $\mathcal{X}$ of R-modules, let us denote by $\mathcal{X}^{[1]}$ the class of all R-modules M with $\operatorname{Ext}^1(\mathcal{X}, M) = 0$, and by ${}^{[1]}\mathcal{X}$ that of all M with $\operatorname{Ext}^1(M, \mathcal{X}) = 0$ (the cotorsion literature prefers to write $\mathcal{X}^{\perp}$ and ${}^{\perp}\mathcal{X}$ instead of $\mathcal{X}^{[1]}$ and ${}^{[1]}\mathcal{X}$, respectively, but this notation is in conflict with other conventions concerning the use of the symbol $^{\perp}$, thus we have avoided to use the symbol $^{\perp}$ altogether; on the other hand, one may extend our notation to deal with an arbitrary interval I of natural numbers, writing say $\mathcal{X}^I$ for the class of all R-modules M with $\operatorname{Ext}^i(\mathcal{X}, M) = 0$ for all $i \in I$). A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is said to be a *cotorsion pair* provided $\mathcal{A} = {}^{[1]}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{[1]}$, and in this case, the intersection $\mathcal{A} \cap \mathcal{B}$ is called its *kernel.* Of course, starting with an arbitrary class $\mathcal{X}$ of R-modules, the pair $({}^{[1]}\mathcal{X}, ({}^{[1]}\mathcal{X})^{[1]})$ is a cotorsion pair, it is called the cotorsion pair *generated by* $\mathcal{X}$. Similarly, $({}^{[1]}(\mathcal{X}^{[1]}), \mathcal{X}^{[1]})$ is a cotorsion pair, it is called the cotorsion pair *cogenerated by* $\mathcal{X}$.

The starting point for our discussion here is the following result of our paper: Let Λ be a concealed canonical algebra with trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ and let $\mathcal{C} = r(\mathbf{q})$ and $\mathcal{D} = l(\mathbf{t})$. Also, denote by ω the intersection $\mathcal{C} \cap \mathcal{D}$. *The pair* $(\mathcal{C}, \mathcal{D})$ *is a cotorsion pair, and it is generated by* $\mathbf{t}$ *and cogenerated by* $\mathbf{q}$. Namely, the definitions immediately imply that $\mathcal{D} = {}^{[1]}\mathbf{t}$ and $\mathcal{C} = \mathbf{q}^{[1]}$, using the Auslander-Reiten formula, the fact that both $\mathbf{t}$ and $\mathbf{q}$ are closed under the Auslander-Reiten translations as well as that $\mathbf{t}$ consists of modules of projective dimension at most 1, whereas $\mathbf{q}$ consists of modules of injective dimension at most 1. Also, the basic splitting theorem (Theorem 2, or better **9.**(5)) asserts that $\mathcal{C} \subseteq {}^{[1]}\mathcal{D}$. The reverse implication comes for example from Proposition 2, or better **9.**(6): $\mathcal{C} = {}^{[1]}\omega \subseteq {}^{[1]}\mathcal{D}$. Of course, the last reference yields also that *the cotorsion pair* $(\mathcal{C}, \mathcal{D})$ *is both generated and cogenerated by* ω.

Perfectness. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *perfect* provided the class $\mathcal{A}$ is closed under direct limits. Our definition $\mathcal{C} = r(\mathbf{q})$ assures us that *the cotorsion pair* $(\mathcal{C}, \mathcal{D})$ *is perfect,* since $\mathbf{q}$ consists of finitely generated modules.

Completeness. A monomorphism $f\colon M \to M'$ in $\operatorname{Mod} R$ with target M' in $\mathcal{B}$ and cokernel in $\mathcal{A}$ is called a *special* $\mathcal{B}$*-preenvelope*; of course, such a monomorphism is always a left $\mathcal{B}$-approximation; in case it is also minimal, it is said to be a *special* $\mathcal{B}$*-envelope.* An epimorphism $g\colon N' \to N$ with N' in $\mathcal{A}$ and kernel in $\mathcal{B}$ is called a *special* $\mathcal{A}$*-precover*; such an epimorphism is always a right $\mathcal{A}$-approximation; in case it is also minimal, it is said to be a *special* $\mathcal{A}$*-cover.* The Salce paper [S] provides a proof for the following important result: Given a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\operatorname{Mod} R$, then every R-module has a special $\mathcal{B}$-preenvelope if and only if every R-module has a special $\mathcal{A}$-cover. In case special $\mathcal{B}$-preenvelopes (and therefore also special $\mathcal{A}$-precovers exist for all R-modules, the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *complete.* According to Enochs [E], given a complete and perfect cotorsion pair $(\mathcal{A}, \mathcal{B})$, then any R-module has a special $\mathcal{B}$-envelope as well as a special $\mathcal{A}$-cover.

Recall that Theorem 1 (or better **6.**(3)) of our paper provides a minimal left ω-approximation for the modules in $\mathcal{C}$. Since the cokernel of such a map belongs to $\mathcal{C}$, it follows immediately that it even is a minimal left $\mathcal{D}$-approximation, thus a special $\mathcal{D}$-envelope. Of course, given a module in $\mathcal{D}$, the identity map is a special $\mathcal{D}$-envelope. According to Corollary 1 (or better **9.**(2)) every R-module is the direct sum of a module in $\mathcal{C}$ and a module in $\mathcal{D}$, thus every R-module has a special $\mathcal{D}$-envelope. This shows that $(\mathcal{C},\mathcal{D})$ is a complete cotorsion pair. Note that we also have shown the existence of a special $\mathcal{C}$-cover, for any R-module: For the modules in $\mathcal{D}$, this is asserted in Theorem 5 (or better **9.**(4)); for the modules in $\mathcal{C}$ one just takes the identity map.

For the existence of special $\mathcal{D}$-envelopes and special $\mathcal{C}$-covers one also may refer to recent investigations of Eklof and Trlifaj [ET]: they have shown that every cotorsion pair $(\mathcal{A},\mathcal{B})$ generated by a class of algebraically compact modules is both perfect and complete. Now, all the modules in $\mathbf{t}$ are finite dimensional, thus algebraically compact. Also, all the modules in ω are algebraically compact. But we know that the cotorsion pair $(\mathcal{C},\mathcal{D})$ is generated by $\mathbf{t}$ as well as by ω.

Resolutions and Coresolutions. Assume that $(\mathcal{A},\mathcal{B})$ is a perfect and complete cotorsion pair in $\operatorname{Mod} R$, and let $\mathcal{K}=\mathcal{A}\cap\mathcal{B}$. Observe that for any module M in $\mathcal{A}$, there exists an exact sequence

$$0\to M\xrightarrow{d^0} M^0\xrightarrow{d^1} M^1\xrightarrow{d^2}\cdots$$

such that $\operatorname{Im}(d^i)\to M^i$ is a minimal left $\mathcal{B}$-approximation, for all $i\ge 0$; such a sequence is called a *minimal $\mathcal{B}$-coresolution,* it is unique up to isomorphism and all the modules M^i actually belong to $\mathcal{K}$. In order to show the existence, let $d^0\colon M\to M^0$ be a special $\mathcal{B}$-envelope of M, then M^0 is an extension of M by the cokernel of d^0. Now both modules M and $\operatorname{Cok}(d^0)$ belong to $\mathcal{A}$, thus M^0 belongs to $\mathcal{K}$. Since the cokernel $\operatorname{Cok}(d^0)$ belongs to $\mathcal{A}$, we can continue. Dually, for any module N in $\mathcal{B}$, there exists an exact sequence

$$\cdots\xrightarrow{d_2} N_1\xrightarrow{d_1} N_0\xrightarrow{d_0} N\to 0$$

such that $N_i\to\operatorname{Im}(d_i)$ is a minimal right $\mathcal{A}$-approximation, for all $i\ge 0$; such a sequence is called a *minimal $\mathcal{A}$-resolution*; and again, it is unique up to isomorphism and all the modules N_i belong to $\mathcal{K}$. In our setting $(\mathcal{C},\mathcal{D})$ the $\mathcal{D}$-coresolution of any module in $\mathcal{C}$ has been exhibited in Theorem 1 (or better **6.**(3)), it is a short exact sequence with M^1 a direct sum of Prüfer modules; the $\mathcal{C}$-resolution of any module in $\mathcal{D}$ has been exhibited in Theorem 5 (or better **6.**(4)), it is a short exact sequence with N_1 a direct sum of copies of the generic module G.

Now assume in addition that the modules in the kernel $\mathcal{K}=\mathcal{A}\cap\mathcal{B}$ can be classified by invariants. Then one may use the $\mathcal{B}$-coresolutions for the modules M in $\mathcal{A}$ and the $\mathcal{A}$-resolutions for the modules N in $\mathcal{B}$ in order to attach a sequence of invariants to M, or N, respectively. In our case $(\mathcal{C},\mathcal{D})$, we know that any module in $\omega=\mathcal{C}\cap\mathcal{D}$ is a direct sum of copies of the generic module G and of Prüfer modules, and such a direct decomposition is unique up to isomorphism. Let us denote by $\mathcal{S}$ the set of isomorphism classes of indecomposable modules in ω (the letter S stands for "spectrum"); there is a special element, say $s=0$, which corresponds to the generic module, the remaining elements of $\mathcal{S}$ correspond bijectively to the simple objects in $\mathbf{t}$.

For any $s\in\mathcal{S}$ and a module M in ω, we denote by $\mu(s,M)$ the multiplicity of s in a direct decomposition of M. Now, given a module M in $\mathcal{C}$, $s\in\mathcal{S}$ and $i\ge 0$, we may define

$$\mu^i(s,M)=\mu(s,M^i).$$

Note that these invariants are zero for $i\notin\{0,1\}$, and also $\mu^1(0,M)=0$ for all M in $\mathcal{C}$. The invariant $\mu^0(0,M)$ has been called the *rank* of the module M in [R1].

Similarly, given a module N in $\mathcal{D}$, $s\in\mathcal{S}$ and $i\ge 0$, we may define

$$\mu_i(s,N)=\mu(s,N_i).$$

Again, these invariants are zero for $i\notin\{0,1\}$, and also $\mu^1(s,M)=0$ for all M in $\mathcal{C}$ and $s\neq 0$.

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